PHIL 3340 (Symbolic Logic II)
First Exam - Sample Questions
Answer Key

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1 Set Theory

1. Let set $A = \{1, 2, 3\}$ and let $B = \{2, 4, 6, 8\}$. Give a function from A to B that is total and one-to-one but not onto.

   One possibility: $\{(1, 2), (2, 4), (3, 6)\}$.

2. Give a function from B to A that is total and onto but not one-to-one.

   One possibility: $\{(2, 1), (4, 2), (6, 3), (8, 3)\}$.

3. Consider the function $f = \{(1, 2), (2, 4), (3, 4), (4, 5)\}$. Give f^{-1}.

   $f^{-1} = \{(2, 1), (5, 4)\}$. Note that $f^{-1}(4)$ is undefined. (Why?)

2 Enumerability

4. Give an informal proof that the rational numbers are enumerable.

Every rational number is expressable in the form n/d where the numerator, n, and the denominator, d, are both integers. We can arrange all such numbers in a two-dimensional array by letting columns represent numerators and rows represent denominators. (Thus every rational fraction n/d will be represented at row d, column n.) Finally, we can weave through the two-dimensional array so as to convert it into a one-dimensional list.

5. Give an informal proof that the real numbers between 0 and 1 (including 0 and excluding 1) are not enumerable.

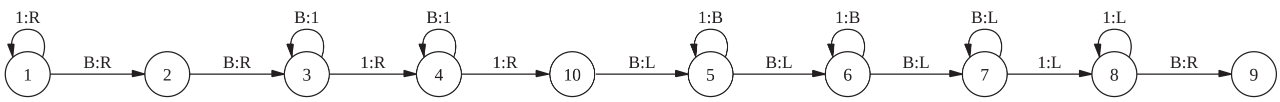
Every real number between 0 and 1 is expressible in decimal notation by a decimal point followed by enumerably many digits. Suppose (for reductio) that they are enumerable. Then they can be arranged in a two-dimensional list, where $list(i, j)$ is the j-th digit of the i-th real number in the list. Now we can construct a new real number as follows. The i-th digit of our new number, new_i, will be $list(i,i)+1$ if $list(i,i) < 9$. (We could represent this more simply by simply writing $new_i = (list(i,i)+1)\%10$, where '%' is the *modulus* operator, defined as follows: for any integers n, m, $n\%m$ is the remainder when n is divided by m.) If $list(i,i) = 9$ then the i-th digit of the new number is 0.

By hypothesis, the new number is represented somewhere in our original list. So for some number j, the new number must be the j-th item in the list. The j-th digit of this number is $list(j,j)$ (by the way we defined the list). But the j-th digit also must be $(list(j,j)+1)\%10$. But that is impossible. (To satisfy both conditions, new_j would have to be both 0 and 1, or both 1 and 2, or both 2 and 3, or . . ., or both 9 and 0.)

Since the assumption that the reals from 0 to 1 are enumerable leads to a contradiction, it must be false.

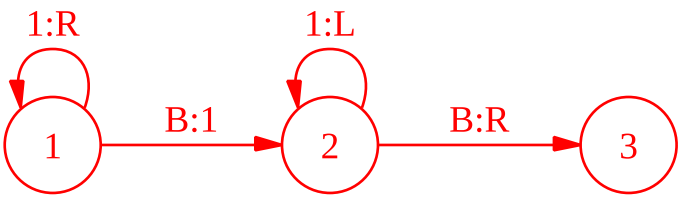
3 Computability

6. Consider the following Turing machine. What function $f : \mathbb{X}^+ \rightarrow \mathbb{X}^+$ does it compute?



$f(x) = x$

7. Write a Turing machine to compute the function $f(n) = n + 1$.



4 Recursive Functions

8. Consider the function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined as follows: $f(x,y) = |x^2 - 2y|$. (That is, f is the absolute value of $x^2 - 2y$.) What is Mn$[f](2)$? How about Mn$[f](3)$? Mn$[f](4)$?

Mn$[f](2) = 2$, Mn$[f](3)$ is undefined, Mn$[f](4) = 8$.