Symbolic Logic II: Sample Final Exam
Answers

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1 Completeness for Propositional Logic

1. Consider a language $\mathcal{L} = \{a, b, Small, Cube\}$. (*Small* and *Cube* should both be understood to be one-place predicates.) Give a truth assignment h for this language.

We need to assign truth values to every atomic sentence. There are sixteen different ways to do this; here is one:

$Small(a) \mapsto True$
$Small(b) \mapsto False$
$Cube(a) \mapsto True$
$Cube(b) \mapsto False$

2. As you know, your function h can be extended to a function $\hat{h}$ which assigns a truth value for every sentence of $\mathcal{L}$. What truth value does $\hat{h}$ assign to the sentence $Small(a) \wedge Cube(a)$? Why?

True, because $\hat{h}$ is an extension of h, and h makes $Small(a)$ and $Cube(a)$ both true. Since (by the clause for conjunction in the definition of truth in a model) a conjunction is true if and only if both conjuncts are true, $Small(a) \wedge Cube(a)$ is assigned true by $\hat{h}$.

3. Define formal completeness. (Remember, this is different from the completeness of a deductive system. Formal completeness is a property of a set of sentences, not a property of a deductive system.)

A set of sentences Γ is formally complete if and only if, for every sentence S in the language, there is a proof, using sentences in Γ as premises, of at least one of S and $\neg S$.

(Of course, if Γ is also formally consistent, then there can't be a proof of both S and $\neg S$, so there must be a proof of exactly one of them.)

4. Is the following set of sentences formally complete? Prove that your answer is correct. $\{Cube(a) \lor Small(a),\ Small(b),\ \neg(Cube(a) \land Small(b)\}$.

No, it is not formally complete. If a set is formally complete, then every atomic sentence or its negation must be provable from the sentences in the set. This set does not contain the sentence $Cube(b)$ anywhere, and it's a consistent set, so we can't deduce either $Cube(b)$ or $\neg Cube(b)$ from the set. So the set is not formally complete.

(More formally, we could find two truth assignments that make all the sentences in the set true, in one of which $Cube(b)$ gets the value True, and in the other of which it gets the value False. This shows that neither $Cube(b)$ nor its negation is a tautological consequence of the sentences in the set. But since our propositional deductive system is sound and complete, a sentence is provable from a set of premises using the propositional rules if and only if it is a tautological consequence of the premises. So neither $Cube(b)$ nor $\neg Cube(b)$ is provable from the sentences in the set.)

5. Prove that $\perp$ is derivable from $\Gamma \cup \{\neg S\}$ using the propositional rules if and only if S is derivable from Γ using the propositional rules. (Notice that this is a biconditional, so you will need to prove both directions.)

Left-to-right direction: Assume that $\perp$ is derivable from $\Gamma \cup \{\neg s\}$. So begin a proof using the sentences in Γ as premises. Start a subproof in which you assume that $\neg S$. Our assumption shows that, inside the subproof, you can derive $\perp$. Do so. Then back in the main proof, we can derive $\neg\neg S$ from the subproof by the rule of $\neg$Intro. Finally, we can derive S from $\neg\neg S$ by using $\neg$Elim.

Right-to-left direction: Assume S is derivable from Γ using the propositional rules. Begin a proof that has all the sentences in Γ together with $\neg S$ as premises. Use the sentences in Γ to derive S (our assumption guarantees we can do this). Now from S, which we've just derived, and $\neg S$, which is one of our premises, we derive $\perp$ by $\perp$Intro.

2 Completeness for First-Order Logic

6. Define what it means for a deductive system for first-order logic to be complete.

To say that a deductive system for first-order logic is complete is to say that for any set of premises Γ and any sentence S, if S is a first-order consequence of Γ, then S is derivable from Γ in the deductive system.

7. What is a "witnessing constant"? What sentences in the Henkin set make use of witnessing constants?

A "witnessing constant" is a constant that provides an instance of an existential formula. It "witnesses" to the truth of the existential sentence. We add to the language, for every formula $F(x)$, a witnessing constant $c_{F(x)}$. Intuitively, the purpose of the constant is to name an object which has the property expressed by F if anything does. So the Henkin set contains, for each formula $F(x)$, the sentence $\exists x F(x) \rightarrow F(c_{F(x)}$.

8. Explain what the Elimination Theorem is.

The Elimination Theorem says that if a sentence S is provable, using only the propositional rules, from the sentences in a set Γ together with the Henkin set H for the language, then S is provable from Γ alone using a full first-order deductive system. (If we use $\vdash_T$ to represent provability from the propositional rules alone, we can symbolize this by saying that if $\Gamma \cup H \vdash_T S$, then $\Gamma \vdash S$.

3 Arithmetization

9. Given an appropriate Gödel numbering, there is a recursive function * (the concatenation function) on the natural numbers such that if an expression e consists of an expression e_1 followed by another expression e_2, and if g is the Gödel number of e, g_1 is the Gödel number of e_1 and g_2 is the Gödel number of e_2, then $g_1 * g_2 = g$. Given this fact, prove that there is a two-place recursive function on the natural numbers that takes the Gödel numbers of two sentences S_1 and S_2 as arguments and returns as its value the Gödel number of their conjunction $(S_1 \wedge S_2)$.

Easy! Let the Gödel number of '(' be $g_($, the Gödel number of $\wedge$ be $g_\wedge$, and the Gödel number of ')' be $g_)$. Then $g_(* g_1 * g_\wedge * g_2 * g_)$ is the function we need. This is just the result of composing four applications of the $*$ function. Since $*$ is recursive, and the composition of two recursive functions is recursive, the function just described is recursive.

4 Representability

10. Here is the first axiom of minimal arithmetic: (Q1) $\forall x(\mathbf{0} \neq x')$. What does it mean?

0 is not the successor of any number.

11. What is a *theory*?

A theory is a set of sentences the contains all of the sentences that are provable from sentences in the set. (It is "closed under derivation.")

12. Is it possible for a theory to be a finite set? Prove your answer.

This one requires a moment's thought. But clearly it cannot be finite. Suppose that the set contains at least one sentence, S. Then, since it contains all the sentences provable from S, it must also contain $S \wedge S$, $(S \wedge (S \wedge S))$, and so on.

But what if S is the empty set? It still must contain the sentence $\forall x(x = x)$, which is provable from no premises at all. But then it must also contain $\forall x(x = x) \wedge \forall x(x = x)$, $(\forall x(x = x) \wedge (\forall x(x = x) \wedge \forall x(x = x)))$, and so on.

13. What does it mean to say that a function f is representable in **Q**?

It means that there is a formula ψ of $\mathcal{L}*$ such that for any natural numbers a and b, $f(a) = b$ if and only if $\psi(\mathbf{a}, \mathbf{b})$ is a theorem of **Q**. (Recall that **a** and **b** are the standard numerals for a and b.)

5 Incompleteness

14. State Gödel's first incompleteness theorem.

For any set of sentences **T** that extends **Q**, if **T** is axiomatizable and consistent, then it is not complete; that is, there is at least one sentence of $\mathcal{L}*$ which is true in the standard interpretation but not contained in **T**.

Another way to say the same thing: for any set of axioms at least as strong as those of minimal arithmetic, if the axioms are consistent, then there is at least one true sentence of $\mathcal{L}$ that is not derivable from the axioms.

15. The Diagonal Lemma says that, for every formula B of $\mathcal{L}*$, and every theory **T** that extends **Q**, there is a sentence G with Gödel numeral **g** such that $\mathbf{T} \vdash G \leftrightarrow B(\mathbf{g})$. Use the Diagonal Lemma to prove that, if **T** is consistent, there is no formula θ that is provably satisfied by all and only the Gödel numbers of sentences in **T**.

Suppose that there were such a formula θ. By hypothesis, $\theta(x)$ would be satisfied by all and only the Gödel numbers of sentences of $\mathcal{L}*$ that are true in the standard interpretation. But also, by the Diagonal Lemma, since if θ is a formula so is $\neg\theta$, there is a sentence G such that $\mathbf{T} \vdash G \leftrightarrow \neg\theta(\mathbf{g})$, where **g** is the Gödel number of G.

Suppose that G is not a theorem of **T**. Since all and only the Gödel numbers of theorems of **T** satisfy the formula $\theta(x)$, g does not satisfy $\theta(x)$. Since we can prove, for every number, whether it satisfies θ or not, $\mathbf{T} \vdash \neg\theta(\mathbf{g})$. But then we can use biconditional elimination to prove G, which means that it is a theorem. Contradiction! Since the assumption that G is not a theorem leads to a contradiction, we conclude that it is a theorem.

Since G is a theorem, its Gödel number satisfies θ. But then, since we are supposing that we can prove for every number whether it satisfies θ or not, we can prove $\theta(\mathbf{g})$. But since $\mathbf{T} \vdash G \leftrightarrow \neg\theta(\mathbf{g})$, we can use this to prove $\neg G$. And of course, to say G is a theorem is to say we can prove it. So from **T** we can prove both G and $\neg G$. So **T** is inconsistent. This contradicts our assumption that **T** is consistent.

So the assumption that there is a formula θ that is provably satisfied by all and only the Gödel numbers of sentences in **T** leads to a contradiction, and must be false.

16. Use the result of problem 15 to prove that, if Church's Thesis is true, then there is no decision procedure for determining, for any sentence S, whether $S \in \mathbf{T}$.

For every recursive function, there is a formula that represents that function in **Q** (and hence, there is such a formula in any set of sentences **T** that extends **Q**). Since there is no formula that represents a function that is true of all and only the Gödel numbers of sentences in **T** (by problem 15 above), there is no recursive function that picks out all and only the Gödel numbers of sentences in **T**. If Church's Thesis is correct, then every computable function is recursive. So, if Church's Thesis is correct, we cannot compute a function that determines all and only the Gödel numbers of sentences in **T**. If there were a decision procedure determining, for any sentence S, whether it were in **T**, then we would be able to combine that with our Gödel numbering function to compute the Gödel numbers of sentences in **T**. So there can't be a decision procedure determining which sentences are in **T**.