Symbolic Logic II: Sample Final Exam

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1 Completeness for Propositional Logic

1. Consider a language $\mathcal{L} = \{a, b, Small, Cube\}$. ($Small$ and $Cube$ should both be understood to be one-place predicates.) Give a truth assignment h for this language.

2. As you know, your function h can be extended to a function $\hat{h}$ which assigns a truth value for every sentence of $\mathcal{L}$. What truth value does $\hat{h}$ assign to the sentence $Small(a) \wedge Cube(a)$? Why?

3. Define formal completeness. (Remember, this is different from the completeness of a deductive system. Formal completeness is a property of a set of sentences, not a property of a deductive system.)

4. Is the following set of sentences formally complete? Prove that your answer is correct. $\{Cube(a) \vee Small(a),\ Small(b),\ \neg(Cube(a) \wedge Small(b)\}$.

5. Prove that $\perp$ is derivable from $\Gamma \cup \{\neg S\}$ using the propositional rules if and only if S

is derivable from T using the propositional rules. (Notice that this is a biconditional, so you will need to prove both directions.)

2 Completeness for First-Order Logic

6. Define what it means for first-order logic to be complete.

7. What is a "witnessing constant"? What sentences in the Henkin set make use of witnessing constants?

8. Explain what the Elimination Theorem is.

3 Arithmetization

9. Given an appropriate Gödel numbering, there is a recursive function * (the concatenation function) on the natural numbers such that if an expression e consists of an expression e_1 followed by another expression e_2, and if g is the Gödel number of e, g_1 is the Gödel number of e_1 and g_2 is the Gödel number of e_2, then $g_1 * g_2 = g$. Given this fact, prove that there is a two-place recursive function on the natural numbers that takes the Gödel numbers of two sentences S_1 and S_2 as arguments and returns as its value the Gödel number of their conjunction $(S_1 \wedge S_2)$.

4 Representability

10. Here is the first axiom of minimal arithmetic: (Q1) $\forall x(\mathbf{0} \neq x')$. What does it mean?

11. What is a *theory*?

12. Is it possible for a theory to be a finite set? Prove your answer.

13. What does it mean to say that a function is representable in $\mathbf{Q}$?

5 Incompleteness

14. State Gödel's first incompleteness theorem.

15. The Diagonal Lemma says that, for every formula B of $\mathcal{L}*$, and every theory $\mathbf{T}$ that extends $\mathbf{Q}$, there is a sentence G with Gödel numeral $\mathbf{g}$ such that $\mathbf{T} \vdash G \leftrightarrow B(\mathbf{g})$. Use the Diagonal Lemma to prove that there is no formula θ that is satisfied by all and only the Gödel numbers of sentences in $\mathbf{T}$.

16. Use the result of problem 15 to prove that, if Church's Thesis is true, then there is no decision procedure for determining, for any sentence S, whether $S \in \mathbf{T}$.