

Deduction: Notes on Leary, Chapter 2

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1 Introduction

In Chapter 2, Leary develops a deductive system. The aim here is to develop a deductive system that is independent of the semantics developed in chapter 1. A deductive system will specify the conditions under which a sequence of formulas in a language \mathcal{L} counts as a deduction (also known as a derivation or a proof).

Leary's system is different in a number of ways from Barwise and Etchemendy's (or those in other basic symbolic logic texts written by philosophers). Some of the differences of detail will emerge below, as we examine Leary's system. But a few points are worth mentioning immediately.

1. Unlike B&E's system, Leary offers an *axiomatic* system. That is, in addition to rules of inference, Leary has axioms which count as part of the formal system.
2. Like most mathematicians, and unlike most philosophers who write about logic, Leary is not much concerned with evaluating natural-language arguments. Philosophers are typically interested in determining under what conditions a conclusion can be said to follow from a collection of *premises*. In general, these premises will be contingent truths, things that may be true but could have been false. For instance, we may want to know whether the sentence $\text{Tet}(a)$ is a consequence of the premises $(\forall x)(\text{Large}(x) \rightarrow \text{Tet}(x))$ and $\text{Large}(a)$. Now, it is certainly not necessary in any sense that either of these premises is true, but we can still investigate whether the conclusion can be deduced from the premises. Mathematicians tend not to worry about cases like this. Their concern is mathematics, where all the sentences of interest are either necessarily true or necessarily false. Thus, when Leary discusses the notion of a deduction of ϕ from Σ , written $\Sigma \vdash \phi$, he describes Σ as a set of "nonlogical axioms". This is a good term to describe, say, the axioms of set theory or number theory, or even sentences such as $(\forall x)(\forall y)(\text{Larger}(x, y) \rightarrow \text{Smaller}(y, x))$. But it's not a good term to describe the premises of an argument if these premises are not necessary truths.

This is only a terminological point, however; the system will work fine whether Σ consists of nonlogical axioms or of other sentences we might want to use as premises.

It's just that a class of cases that is extremely important to philosophers seems to rather drop out of view for mathematicians.

3. Also, Leary develops a system within which we can talk about deductions of or from formulas that are not sentences. Most logic texts in philosophy limit themselves to discussing deduction as a relation between sentences. (I'm not sure the added generality of including nonsentential formulas is worth the trouble, any more than I'm sure it's worth the added complication it causes in the semantics to define validity and logical implication for nonsentential formulas. But hey, who am I to argue?)

We can begin with Leary's formal definition of a deduction in terms of axioms and rules of inference, and then investigate which specific axioms and rules of inference we will need. Here, then, is Leary's Definition 2.2.1:

Definition: If Σ is a set of \mathcal{L} -formulas and D is a finite sequence $\langle \phi_1, \phi_2, \dots, \phi_n \rangle$ of \mathcal{L} -formulas, then D is a **deduction from** Σ if and only if for each i , $1 \leq i \leq n$, either

1. $\phi_i \in \Lambda$ (ϕ_i is a logical axiom), or
2. $\phi_i \in \Sigma$ (ϕ_i is a nonlogical axiom), or
3. There is a rule of inference $\langle \Gamma, \phi_i \rangle$ such that $\Gamma \subseteq \{\phi_1, \phi_2, \dots, \phi_{i-1}\}$.

So Σ will be a set of premises (which may or may not be usefully called "nonlogical axioms"; see earlier comment) and Λ will be a set of logical axioms. Notice that *rules of inference* are characterized as ordered pairs $\langle \Gamma, \phi \rangle$, where Γ is a set of formulas and ϕ is a formula. So for instance, in Barwise and Etchemendy's version of propositional logic, $\langle \{\phi, \phi \rightarrow \psi\}, \psi \rangle$ would be the rule of inference Conditional Elimination, and $\langle \{\phi\}, \phi \vee \psi \rangle$ would be the rule Disjunction Introduction.

Some of B&E's rules are difficult to fit into this format. Leary's deductive system does not include the idea of subproofs, so rules like Conditional Introduction and Disjunction Elimination are tricky to state. For Disjunction Elimination the following legitimates the same inferences as B&E's rule: $\langle \{\phi \vee \psi, \phi \rightarrow \theta, \psi \rightarrow \theta\}, \theta \rangle$. I don't think that there is a way to squeeze B&E's rule of Conditional Introduction into this format. (One way to express their rule is to say that if $\Sigma \cup \Lambda \cup \{\phi\} \vdash \psi$, we may write down $\phi \rightarrow \psi$. But this does not fit the $\langle \Gamma, \phi \rangle$ format.) However, we could replace their rule by another that accomplishes the same thing and can be expressed in the desired format; for instance, the rule $\langle \{\neg\phi \vee \psi\}, \phi \rightarrow \psi \rangle$. Alternatively, we could do the same sort of thing Leary does when specifying propositional rules: we could say that if $\Sigma \cup \Gamma \cup \phi \vdash \psi$, then $\langle \Gamma, \phi \rightarrow \psi \rangle$ is a rule of inference.

2 Propositional Logic

Leary does not waste much time on the propositional portion of logic. He offers us a single definition to provide propositional rules of inference. However, this single definition in effect gives us an infinite number of propositional rules!

Here is the definition; then a few comments.

Definition: If Γ is a finite set of \mathcal{L} -formulas, ϕ is an \mathcal{L} -formula, and ϕ is a propositional consequence of Γ , then $\langle \Gamma, \phi \rangle$ is a **rule of inference of type (PC)**.

Here is the idea. ϕ is a propositional consequence of Γ just in case it is what B&E call a “tautological consequence” of Γ , that is, if there is no row of a joint truth table in which every sentence in Γ is T and ϕ is F. (Notice that Leary’s procedure on p. 59 for converting a formula β of first-order logic into a formula β_P of propositional logic amounts to the same thing as B&E’s procedure for identifying the truth-functional form of a quantifier sentence on p. 261 of *Language, Proof, and Logic*.) So the above definition gives us infinitely many propositional rules of inference. These include most of B&E’s rules, although as noted above it will not give us \rightarrow Intro or \vee Elim. But there will be more than enough other rules to make up for this lack!

And of course the rules we get in this way will be massively redundant. For instance, we will have the rules $\langle \{\neg\phi\}, \phi \rightarrow \psi \rangle$ and $\langle \{\psi\}, \phi \rightarrow \psi \rangle$ and $\langle \{\neg(\phi \wedge \neg\psi)\}, \phi \rightarrow \psi \rangle$, as well as $\langle \{\neg\phi \vee \psi\}, \phi \rightarrow \psi \rangle$.

There is another thing that makes me uncomfortable with this way of handling propositional logic. We are working up to proving some interesting connections between the semantic relation expressed by ‘ \models ’ and the proof-theoretic relation expressed by ‘ \vdash ’. In order for this to be interesting, we need to define each of these relations independently of the other. But truth tables are a semantic device, so defining propositional rules of inference in terms of truth tables makes proving soundness and completeness for the propositional part of logic basically trivial.

On the other hand, soundness and completeness results for propositional logic are fairly straightforward in any case; predicate logic is where the action’s at. So although I think there is a theoretical cost to be paid for treating propositional logic in this way, perhaps it is a fairly minor price.

3 Predicate Logic

For predicate logic, we need several additional axioms and rules. The axioms are all fairly straightforward. We have, first of all, three axioms (actually axiom schemata) for identity:

$$x = x \text{ for each variable } x$$

$$[(x_1 = y_1) \wedge (x_2 = y_2) \wedge \dots \wedge (x_n = y_n)] \rightarrow (f(x_1, x_2, \dots, x_n) = f(y_1, y_2, \dots, y_n))$$

$$[(x_1 = y_1) \wedge (x_2 = y_2) \wedge \dots \wedge (x_n = y_n)] \rightarrow (R(x_1, x_2, \dots, x_n) = R(y_1, y_2, \dots, y_n))$$

Notice that the first axiom corresponds to B&E's rule of =Intro, and the other two axioms correspond to B&E's rule of =Elim. Given the way the quantifier rules work, we will be able to get from $x = x$ to $(\forall x)(x = x)$, and from there to $t = t$ for any term t (subject to certain substitutability requirements), so the first axiom gives results as general as =Intro even though it only explicitly concerns variables. Similarly, the second and third axioms only explicitly allow substitution of variables, and then only into terms consisting of function symbols followed by terms, and into relation symbols followed by terms. In fact, however, the effect will be to allow substitution of any term t_2 for any term t_1 if we have the identity statement $t_1 = t_2$, and to allow this in any formula.

We also have two quantifier axioms. These are also quite straightforward.

$$(\forall x\phi) \rightarrow \phi_t^x, \text{ if } t \text{ is substitutable for } x \text{ in } \phi.$$

$$\phi_t^x \rightarrow (\exists x\phi), \text{ if } t \text{ is substitutable for } x \text{ in } \phi.$$

The first of these quantifier axioms corresponds to the rule \forall Elim, and the second corresponds to the rule \exists Intro. You will recall that of the four quantifier rules in B&E, these are the two that have no restrictions and are easy to understand. To fully understand the axioms, we need to recall some definitions from Chapter 1. ϕ_t^x is the result of substituting t for all free occurrences of x in the formula ϕ . t is substitutable for x provided that it does not contain any free variables that would be bound by quantifiers in ϕ . (Here again we see how complicated everything needs to be in order to accommodate nonsentential formulas.) For instance, if t is the term y , then it is not substitutable for x in the formula $(\forall y)(\text{Large}(y) \rightarrow y \neq x)$. [Note to self: find a better example. Need a reasonable formula which would become unreasonable if the substitution were allowed. This one doesn't really qualify because it can't be true to start with, since free variables are essentially treated as though they were universally quantified.]

This leaves us with two quantifier rules of inference, which we may anticipate will correspond to \forall Intro and \exists Elim, the more complicated quantifier rules in B&E's system (as well as most others). And so they do, but at first glance they are a bit mysterious.

First, we have a rule corresponding to \forall Intro, namely the rule

$$\langle \{\psi \rightarrow \phi\}, \psi \rightarrow (\forall x\phi) \rangle$$

provided that x is not free in ψ . (A minor point: x can be any variable. So it might be better to say that, for any variable v , $\langle \{\psi \rightarrow \phi\}, \psi \rightarrow (\forall v\phi) \rangle$ is a rule of inference.)

We also have a rule corresponding to \exists Elim, namely this:

$$\langle \{\phi \rightarrow \psi\}, (\exists x\phi) \rightarrow \psi \rangle$$

provided, again, that x is not free in ψ .

It requires some thought to see that these rules amount to \forall Intro and \exists Elim, and Leary really doesn't help all that much, as his entire discussion of these two rules consists of about half a page. (He also doesn't have any exercises that require using the rules!)

It may help to look at how these rules might actually be used in a deduction. Let's consider a couple of easy quantifier proofs and compare how they work in B&E's system and in Leary's system. [Actually I only had time to type one of these up for the time being.]

To begin with, suppose we want to prove, from the premises that $(\exists x)\text{Tet}(x)$ and $(\forall x)(\text{Tet}(x) \rightarrow \text{Large}(x))$, the conclusion $(\exists x)\text{Large}(x)$. This should be a fairly straightforward deduction. Here is a proof in B&E's system:

1. $(\exists x)\text{Tet}(x)$	
2. $(\forall x)(\text{Tet}(x) \rightarrow \text{Large}(x))$	
3. $\boxed{a} \text{Tet}(a)$	
4. $\text{Tet}(a) \rightarrow \text{Large}(a)$	\forall Elim: 2
5. $\text{Large}(a)$	\rightarrow Elim: 4, 3
6. $\exists x\text{Large}(x)$	\exists Intro: 5
7. $\exists x\text{Large}(x)$	\exists Elim: 1, 3-6

And now, for comparison, here is a proof in Leary's system:

1. $(\exists x)\text{Tet}(x)$	premise (i.e. $\in \Sigma$)
2. $(\forall x)(\text{Tet}(x) \rightarrow \text{Large}(x))$	premise (i.e. $\in \Sigma$)
3. $((\forall x)(\text{Tet}(x) \rightarrow \text{Large}(x)) \rightarrow (\text{Tet}(x) \rightarrow \text{Large}(x)))$	Q1: 2
4. $\text{Tet}(x) \rightarrow \text{Large}(x)$	prop cons: 2,3
5. $\text{Large}(x) \rightarrow (\exists x)\text{Large}(x)$	Q2: 4
6. $\text{Tet}(x) \rightarrow (\exists x)\text{Large}(x)$	prop cons: 4,5
7. $(\exists x)\text{Tet}(x) \rightarrow (\exists x)\text{Large}(x)$	QR \exists : 6
8. $(\exists x)\text{Large}(x)$	prop cons: 1, 7

That's all for now . . .