Propositional Modal Logic: A Few First Steps

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1 Interpretations for Propositional Logic

We have discussed *structures*, i.e. interpretations for a first-order language. We have seen that a structure or interpretation for a full first-order language includes a universe, a specification of an element of the universe for every constant in the language, and specifications of sets of sequences whose members are elements of the universe for each function symbol and relation symbol.

If we restrict our attention to propositional logic, matters are considerably simpler. In propositional logic, we do not need to worry about interpretations for function symbols, relation symbols, or constants, nor do we need to specify a universe. Instead, all we need to do is assign a truth value to every atomic sentence of the language. Then the standard truth tables will determine, for every compound sentence of the language, what its truth value is based on the truth values of the component atomic sentences.

Thus, for propositional logic, an interpretation for a language amounts to one row of the reference columns of a truth table (provided that we have a column for every atomic sentence in the language). Obviously, then, we can apply other things we know about truth tables to interpretations: for instance, since a truth table for a set of sentences has 2^n rows, where n is the number of atomic sentences that are components of the given set of sentences, we can infer that the number of distinct interpretations for a propositional language is also 2^n, where n is the number of atomic sentences in the language.

Since we are only going to consider propositional modal logic, not quantified modal logic, we can make things easier on ourselves by beginning with this relatively simple notion of the interpretation of a propositional language, rather than with the full-blown notion of an interpretation for a full first-order language.

2 Interpretations for Propositional Modal Logic

We begin with a very simple version of structures or interpretations for propositional modal logic. In the next section we will complicate things slightly. Where a propositional language contains atomic sentences and the logical operators $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, a modal propositional language contains all this plus two additional operators, $\Box$ and $\Diamond$. Like $\neg$, these are one-place operators which attach to a single sentence to form a new compound sentence. So, for any sentence A (either atomic or compound), we also have the compound sentences $\Box A$ and $\Diamond A$.

So, here is the simple version. The building blocks of interpretations for propositional modal logic (and for quantified modal logic, for that matter) are *possible worlds.* For *propositional* modal logic, a possible world can be regarded as simply an assignment of truth values to atomic sentences. (So a possible world in a propositional modal language corresponds to a complete interpretation for a nonmodal propositional language.) We could explicitly represent a world as a mapping from atomic sentences to truth values. For instance, given the language $\{A, B, C\}$ with atomic sentences A, B, and C, one particular possible world w could be represented like this. w: $A \mapsto T, B \mapsto F, C \mapsto T$.

Clearly, a possible world w determines a function from atomic sentences to truth values. This suggests a helpful shorthand notation: we can say, for instance, $w(A) = T$ to indicate that w maps the atomic sentence A to the truth value T.

Now, a structure for propositional modal logic is a set of possible worlds, W. One of these possible worlds, represented as $w*$, is singled out as the *actual world.* Hopefully the intuitive motivation for these ideas is clear. Consider a language containing the atomic sentences Heads, Tails representing the ways a particular coin could fall on a particular toss. Then in one possible world, w_1, we will have $w_1(\mathit{Heads}) = T$ and $w_1(\mathit{Tails}) = F$. In another possible world, w_2, we have $w_2(\mathit{Heads}) = F$ and $w_2(\mathit{Tails}) = T$. After we toss the coin, one of these possible worlds will be actual. For instance, if the coin comes up tails, then $w* = w_2$. The other possible world, w_1, although not *actual*, is still *possible*, in the sense that, although the coin actually came up tails, it *could have* come up heads instead.

Of course, for all I have said so far, there are also the worlds w_3 and w_4 such that $w_3(\mathit{Heads}) = T$, $w_3(\mathit{Tails}) = T$, $w_4(\mathit{Heads}) = F$, $w_4(\mathit{Tails}) = F$. We clearly don't want these to actually count as possible worlds, however! How to rule them out? One approach would be to state axioms that rule them out, for example by taking as axioms the propositions (1) (Heads $\vee$ Tails), and (2) $\neg$(Heads $\wedge$ Tails). w_3 fails to make (2) true, and w_4 fails to make (1) true, so they are ruled out by these axioms.

We now need to define truth for sentences in a propositional modal language. Mostly this is exactly the same as for an ordinary propositional language. Possible worlds correspond to interpretations for nonmodal propositional languages. So for any possible world, we can determine the truth values of nonmodal compound sentences by means of the standard truth-table interpretations of the connectives. For example, for any world w, $w(\neg A) = T$ if and only if $w(A) = F$; $w(A \wedge B) = T$ if and only if $w(A) = T$ and $w(B) = T$, and so on. This gives us what we might call truth in a possible world. Truth, period, is simply truth in the actual world. So an atomic sentence A is true, period, if and only if $w * (A) = T$. (To be a bit more formal we could speak of truth *simpliciter* instead of truth, period.)

Where we need the other possible worlds is in the evaluation of modal sentences. For any sentence A, $\Box A$ is true if and only if $(\forall w \in W)(w(A) = T)$, i.e. if and only if *every* possible world in W assigns A the truth value T. Similarly, for any sentence A, $\Diamond A$ is true if and only if $(\exists w \in W)(w(A) = T)$, i.e. if there is at least one world in W that assigns T to A.

We can say that $\Box A$ is true if and only if A is true in every possible world, while $\Diamond A$ is true if and only if A is true in some possible world.

3 Adding the Accessibility Relation

If all we wanted was a modal logic for dealing with metaphysical necessity and possibility, what we have so far might do nicely as a semantics. However, for various other purposes (more on this later) it is nice to have more flexibility. This is achieved by adding another element to a structure or interpretation for propositional modal logic. The new element is an *accessibility relation* R. This is a relation that holds between possible worlds. So, for possible worlds w_1, w_2, to say $R(w_1, w_2)$ is to say that w_2 is accessible from w_1.

The accessibility relation makes no difference to nonmodal sentences. We do need to revise our definitions of truth for modal sentences, however. Now we will say that $\Box A$ is true in a world w if and only if A is true in every world *that is accessible from* w. That is, $w(\Box A) = T$ if and only if $(\forall w' \in W)(R(w, w') \rightarrow w'(A))$. Similarly, $w(\Diamond A) = T$ iff $(\exists w' \in W)(R(w, w') \wedge w'(A) = T)$. As with nonmodal sentences, truth *simpliciter* is just truth in the actual world $w*$.

Pretty clearly, if every world is accessible from every other world, then interpretations that include the accessibility relation will give the same results as interpretations that do not. So our simple version of interpretations for propositional modal logic corresponds to having an accessibility relation R that is *reflexive* (every world is accessible from itself), *symmetric* (for any worlds w_1, w_2, if w_1 is accessible from w_2 then w_2 is accessible from w_1), transitive $\forall w_1 \forall w_2 \forall w_3((R(w_1, w_2) \wedge R(w_2, w_3)) \rightarrow R(w_1, w_3))$, and *euclidean*: $\forall w_1 \forall w_2 \forall w_3((R(w_1, w_2) \wedge R(w_1, w_3)) \rightarrow R(w_2, w_3))$. Put all these conditions together, and you get the result that every world is accessible from every other world.

At this point things get extremely interesting. Each of these conditions on the accessibility relation corresponds to a particular sentence of propositional modal logic, as follows.

$A \to \Box\Diamond A$	*symmetry*: if A is true in a world w, then in every world w' accessible from w, there is a world accessible from w' in which A is true. Since A is true in w, this is guaranteed to be true provided that w is accessible from every world that is accessible from it, which is just what symmetry says.
$\Box A \to \Box\Box A$	*transitivity*: $\Box A$ is true at a world w just in case A is true at every world w' accessible from w. So $\Box\Box A$ is true at a world w just in case A is true at every world accessible from every world accessible from w. Think of the accessibility relation as determining a tree with worlds as nodes. Then transitivity guarantees that any world to which there is a path from w is also directly accessible from w. In that case being true at every world adjacent to w suffices to guarantee being true at every world to which there is a path from w.
$\Diamond A \to \Box\Diamond A$	*euclidean.* $\Diamond A$ is true at a world w iff A is true at some world accessible from w. $\Box\Diamond A$ is true at a world w iff, for every world w' accessible from w, there is a world w'' accessible from w' at which A is true. Notice that the euclidean property guarantees that this will be true. Suppose A is true at a world accessible from w. If that world is accessible from every other world accessible from w, then it will be true that for every world accessible from w, there is an accessible world in which A is true.
$A \to \Diamond A$	*reflexivity*: if every world is accessible to itself, then any world in which A is true will be a world from which there is an accessible world in which A is true.

4 Extensions

The accessibility relation may seem like more trouble than it's worth, and for some purposes this is true. However, it is interesting to think of a variety of other logics as essentially modal logics with a restricted universe of worlds and perhaps a different accessibility relation. For example, consider *deontic logic*, the logic of obligation and permission. If T is the proposition that you always tell the truth, then let $\bigcirc T$ represent the proposition that it is obligatory that you always tell the truth. Just as we could define possibility in terms of necessity, so that $\Diamond A$ is defined as $\neg\Box\neg A$, we can define permissibility in terms of obligatoriness. Let C be the proposition that you eat chocolate ice cream. We can express the idea that this is permissible by saying $\neg \bigcirc \neg C$.

We can think of obligatoriness as truth in all morally perfect worlds, and permissibility as truth in some morally perfect world. But we can't simply restrict our universe to morally perfect worlds, or we will leave the actual world out! So it seems the natural thing to say is that we have a universe W including all metaphysically possible worlds, but from any world, the accessible worlds are all and only the morally perfect worlds. In that case transitivity and the euclidean property will hold, but reflexivity and symmetry will not hold.

Similarly, we can think of always being true, being known to be true, being physically necessary, and so on, as modal notions with modified accessibility relations.