Set Theory Handout

Curtis Brown

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1 Additions to First Order Logic

We began by discussing naive set theory. This is elegant, simple, intuitive, and generally lovely; the only problem is that it leads straight to disaster.

To get naive set theory, we make some small additions to first-order logic. First, we either add a predicate Set() to indicate that something is a set, or we distinguish between two styles of variables, one for anything at all and the other exclusively for sets. Our text chose the latter route, distinguishing between the variables $x, y, z, ...$ which can range over anything at all, and the variables $a, b, c, ...$ which range only over sets.

Next, we add a single new relation symbol to the language of first-order logic, namely the symbol $\in$ to represent set membership. Thus a sentence such as $c \in S$ indicates that the element c is a member of the set S.

Third, we add axioms concerning the logical properties of the set membership relation. In naive set theory, we need to add only two axioms (or more precisely, one axiom and an axiom scheme).

Axiom 1.1 (Axiom of Extensionality) $\forall a \forall b[\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b]$

Axiom 1.2 (Axiom of Comprehension) $\exists a \forall x[x \in a \leftrightarrow P(x)]$

(Note that in the Axiom of Comprehension, $P(x)$ can be replaced by any formula expressible in first-order logic that contains x as a free variable. Also note that $P(x)$ may also contain additional free variables $z_1, z_2, \ldots, z_n$, in which case the instance of the axiom schema must be prefixed by universal quantifiers $\forall z_1, \forall z_2, \ldots, \forall z_n$.)

2 Defining Additional Set-Theoretic Notation

It is a striking fact that these minimal additions to first-order logic allow us to define the rest of the notation of set theory, so that we need not add any more primitives to the language itself.

For example, we can define some familiar set-theoretic ideas as follows.

Definition 1 (List Notation) $\{x_1, x_2, \ldots, x_n\}$ *denotes a set s if and only if* $\forall y(y \in s \leftrightarrow (y = x_1 \lor y = x_2 \lor \ldots \lor y = x_n))$

We could add a clause specifying that this set must be unique (i.e. that for any set s', if $y \in s'$ if and only if it is x_1 or x_2 or ... or x_n, then $s' = s$. However, the Axiom of Extensionality guarantees that this will be true whenever the definition above is satisfied, so we do not need to add this explicitly.

Definition 2 (Brace Notation) $\{x|P(x)\}$ *denotes a set s if and only if*
$\forall z(z \in s \leftrightarrow P(z)) \land \forall b(\forall z(z \in b \leftrightarrow P(z)) \rightarrow b = s)$

This time we have listed both the existence and uniqueness requirements. Notice that the first conjunct says that there is a set consisting of all the items x of which $P(x)$ is true. The Axiom of Comprehension guarantees that there will be such a set for every property $P(x)$ expressible in first order logic, so it guarantees that there will be a set that satisfies the first conjunct. The second conjunct says that there is no more than one set that satisfies the property $P(x)$. The Axiom of Extensionality guarantees that this conjunct will be satisfied whenever the first conjunct is. So, taken together, the two axioms guarantee that there will always be a set that satisfies the conditions for being denoted by $\{x|P(x)\}$, thus guaranteeing that brace notation always succeeds in identifying a unique set.

We can give similar definitions for the intersection and union functions:

Definition 3 (Intersection) $\forall a \forall b \forall z(z \in a \cap b \leftrightarrow (z \in a \land z \in b))$

Definition 4 (Union) $\forall a \forall b \forall z(z \in a \cup b \leftrightarrow (z \in a \lor z \in b))$

(We have simply written these definitions as biconditionals. We could regard these biconditionals as additional axioms of our formal system, but to keep the system simple we will instead regard the right-hand side of the biconditional as defining the left-hand side. The definitions of list and brace notion could be written in this way as well.)

We can also define the relations of subset and proper subset.

Definition 5 (Subset) $\forall a \forall b(a \subseteq b \leftrightarrow \forall x(x \in a \rightarrow x \in b))$

Definition 6 (Proper Subset) $\forall a \forall b(a \subset b \leftrightarrow (\forall x(x \in a \rightarrow x \in b) \land \exists y(y \in b \land y \notin a)))$

Here '$y \notin a$' is simply a helpful abbreviation of the negation '$\neg(y \in a)$'.

3 Naive Set Theory Is Inconsistent

Russell's paradox shows that naive set theory is inconsistent. Not a good thing! This means you can derive contradictions, and of course from a contradiction you can derive anything, so that the system becomes completely worthless.

The text discusses a version of the paradox that involves the issue of whether a set can include its own powerset. But there are also simpler examples. The Axiom of Comprehension says that for any predicate formulable in first-order logic (plus the set membership relation $\in$), there is a set consisting of all the items that satisfy that formula. Well, here is a formula of first-order logic plus membership:

$$x \notin x$$

So we have, as an instance of the Comprehension Axiom,

$$\exists c \forall x (x \in c \leftrightarrow x \notin x)$$

But from this we can derive a contradiction. The above instance says that there is a set that is a member of itself if and only if it is not a member of itself. Call this set *pdox* (since it's a paradoxical set!). (In effect we are setting up an existential elimination subproof.) Then we have $\forall x(x \in pdox \leftrightarrow x \notin x)$. If this is true of every object, then in particular it must be true of *pdox* (by universal elimination). So now we have:

$$pdox \in pdox \leftrightarrow pdox \notin pdox$$

From here we can quickly derive a contradiction. We can prove that for any proposition P, $P \vee \neg P$. So in particular

$$pdox \in pdox \vee pdox \notin pdox$$

For disjunction elimination, assume that $pdox \in pdox$. Then by biconditional elimination, $pdox \notin pdox$. Now assume the other disjunct, $pdox \notin pdox$. A step of Reiteration allows us to write this down again as a conclusion inside our subproof. So we can complete the disjunction elimination step and conclude that $pdox \notin pdox$.

So far so good, but now we can use biconditional elimination again to conclude that $pdox \in pdox$. So we have derived *both* $pdox \in pdox$ and $pdox \notin pdox$ from our starting premises.

Recall that the sentence $pdox \notin pdox$ just abbreviates $\neg(pdox \in pdox)$. So, in Barwise and Etchemendy's system, we can derive $\perp$ by the $\perp$ Intro rule. This does not contain the constant *pdox* that we used to instantiate the existential quantifier, so we can complete our existential elimination subproof and reach $\perp$ as our overall conclusion.

But this is clearly a disaster! After all, by $\perp$ Elim we can derive absolutely anything from $\perp$. So naive set theory is inconsistent.

Notice that the Axiom of Extensionality played no role in this derivation. We used only the rules of first-order logic plus the Comprehension Axiom. Since we know that first-order logic itself is sound, and therefore consistent, the source of the trouble appears to be the Comprehension Axiom.

(What if we restricted the predicates $P(x)$ in the Comprehension Axiom to properties expressible in first-order logic /emphwithout set membership? After all, the problem instance just discussed involves a property that includes the set membership relation. However, that would remove too much power. For instance, we need to be able to prove the existence of the powerset of a set, i.e. the set defined by the property $x \subseteq z$.)

4 Zermelo-Frankel Set Theory

This is just an extremely quick overview. The Axiom of Extensionality remains the same, but the Comprehension Axiom is replaced by something of a hodgepodge of axioms that are not as intuitively obvious. Here is a quick list:

Axiom 4.1 (Axiom of Extensionality) $\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b]$

Axiom 4.2 (Axiom of Separation) $\forall a \exists b \forall x [x \in b \leftrightarrow (x \in a \wedge P(x))]$

Axiom 4.3 (Unordered Pair Axiom) $\forall z_1 \forall z_2 \exists a \forall x (x \in a \leftrightarrow (x = z_1 \vee x = z_2))$

For any objects z_1 and z_2, there is a set whose elements are exactly those objects. (I.e. we can take any two objects in our domain and make a set out of them.) Notice that we get for free the result that for any single object there is a set whose only element is that object: this is simply the special case in which $z_1 = z_2$. A set whose only element is an object a is called the *singleton* of a.

Axiom 4.4 (Union Axiom) $\forall a \exists b \forall x [x \in b \leftrightarrow \exists c \in a (x \in c)]$

This one looks a little strange. The idea is that for any set a of sets, we can construct the union b of all the sets in a. (If we just regard union as a binary operator, it might be more natural just to say that for any two sets there is the union of those sets: $\forall a \forall b \exists c \forall x (x \in c \leftrightarrow (x \in a \vee x \in b))$.

Axiom 4.5 (Powerset Axiom) $\forall z \exists a \forall x (x \in a \leftrightarrow x \subseteq z)$

Axiom 4.6 (Axiom of Infinity) $\exists a (\emptyset \in a \wedge \forall x (x \in a \rightarrow s(x) \in a))$

The function $s(x)$ needs comment here. There are two ways to look at this. We can think of $s(x)$ as the *singleton* function, a function which takes any item as argument and yields the singleton set with that item as its only member as its value. Thus the set that the Axiom of Infinity declares to exist is the set:

$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$$

However, recall that we are also using sets to model the natural numbers. $\emptyset$ is interpreted as 0, $\{\emptyset\}$ as 1, and so on. So we could equally well think of $s(x)$ as the *successor* function which takes any natural number as argument and yields the successor of that number as value.

Axiom 4.7 (Axiom of Replacement)

$$\forall a[\forall x \in a \exists ! y P(x,y) \to \exists b(b = \{y | \exists x \in a P(x,y)\}]$$

Here we have used the informal brace notation for the set b. The idea is that if P(x,y) corresponds to a function from members of a given set to objects, then we can form a new set by replacing every member of the given set with the object the function maps it to.

Axiom 4.8 (Axiom of Choice) *If f is a function with non-empty domain a and for each $x \in a$, $f(x)$ is a non-empty set then there is a function g also with domain a such that for each $x \in a$, $g(x) \in f(x)$.*

This is Barwise and Etchemendy's formulation on p. 436. I haven't even tried to do this entirely in symbols! The general idea is that for any collection of sets, you can choose one member from each of them. The function $f(x)$ gives us our initial collection of sets (namely the values of $f(x)$ for the elements in its domain), and the function $g(x)$ is the function that chooses a single item from each of the sets that are the values of $f(x)$. The Axiom of Choice, although widely accepted, is controversial. There's a very interesting web page on the Axiom of choice, with links to further online material, at http://www.math.vanderbilt.edu/ schectex/ccc/choice.html.

Axiom 4.9 (Axiom of Regularity) $\forall b[b \neq \emptyset \to \exists y \in b(y \cap b = \emptyset)]$

This rules out certain nasty sets, especially certain sets that contain themselves as members. Barwise and Etchemendy mention in particular the set $\{\{\{\ldots\}\}\}$ — this set has only one member, which is the same as itself. (And its member has only member, also identical to itself, and so on! A somewhat unsettling set.) But it violates the Axiom of Regularity, since its intersection with its only member is not empty. In fact any set whose only member is itself, i.e. any set a such that $a = \{a\}$, will violate the Axiom of Regularity.

It may seem at first that the Axiom of Regularity does not rule out all sets that contain themselves as members. Consider the set a such that $a = \{\{5\}, a\}$. The intersection of a with its member listed second is clearly not $\emptyset$, but its intersection with $\{5\}$ is still $\emptyset$, since the only member of $\{5\}$ is 5, and neither of the members of a is 5. So the axiom does not *appear* to rule out *all* sets that contain themselves, only the sets that contain *only* themselves.

However, in combination with the Unordered Pair Axiom, the Axiom of Regularity does in fact rule out *every* set that contains itself as a member. Remember that the Unordered Pair Axiom says that, for every pair of items z_1 and z_2, there is a set containing those items and nothing else. Notice that this axiom includes, as a degenerate case, the case in which $z_1 = z_2$. Therefore, if our set a which contains iself is a set, then there must be a set $\{a\}$ that contains a and nothing else. But we have already seen that the Axiom of Regularity rules this out. So these two axioms together show that there is no set that is a member of itself.