MATH 3355 Non-Euclidean Geometries 03/99

Homework 5 Sans Figures

Chapter 3

Exercises on Betweenness, Congruence and Continuity.

Review Exercises 2, 3, 6, 7, 8, 10, 12, 14, 15, 17 and 18 are true. The rest are false.

7 a We define a Dedekind cut on the ray r as a pair of nonempty disjoint subsets $\{\Sigma_1, \Sigma_2\}$ of r whose union is all of r and with the property that no point of either set is between two points of the other set. We now have to prove that if Dedekind cuts of the lines of our geometry are given by points (that is, if our geometry satisfies the Dedekind Continuity Axiom), then so are Dedekind cuts of any ray r. Precisely, we want to show that given a Dedekind cut $\{\Sigma_1, \Sigma_2\}$ of the ray r there exists a unique point O of r with the property that O is between any pair of points (distinct from O, of course) coming, one from Σ_1 and the other from Σ_2. To that effect we assume that r emanates from A, and that $A \in \Sigma_1$. Let l be the unique line containing the ray r. If we let $\Sigma_1^{'} = \Sigma_1 \cup (\{l\} \setminus r)$ and $\Sigma_2^{'} = \Sigma_2$ then the pair $\{\Sigma_1^{'}, \Sigma_2^{'}\}$ is a Dedekind cut of l.

To see this, assume that $P * Q * R$. First of all, if $P, Q \in \Sigma_2^{'}$ then obviously $P, Q \in \Sigma_2$ so $R \in \Sigma_2^{'}$ because $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of r. On the other hand, if $P, Q \in \Sigma_1^{'}$, several cases may occur: (i) $P, Q \notin r$. Then, without loss of generality, $P * Q * A$, so, by Prop. 3.3, $P * R * A$, i.e. $R \notin r$ and $R \in \Sigma_1^{'}$; (ii) $P, Q \in r$. Then $P, Q \in \Sigma_1$ and $R \in {\Sigma_1}'$ because $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of r; and (iii) $P \notin r$, $Q \in r$. Then either $R = A$ so $R \in \Sigma_1^{'}$, or $P * R * A$ so $R \notin r$ and $R \in \Sigma_1^{'}$, or $A * R * Q$ so $R \in \Sigma_1^{'}$ because $R \in \Sigma_1$ as $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of r. We have covered all possibilities.

Now, by Dedekind's Continuity Axiom, there exists a point O in l giving the cut $\{\Sigma_1^{'}, \Sigma_2^{'}\}$ of l. This point O lies on r, for otherwise we would have points $B \in \Sigma_2$ satisfying $B * O * A$, contradicting the fact that $\Sigma_2 \subset r$. Furthermore, it is obvious that the point O satisfies the requisite properties with respect to pairs of points in Σ_1 and Σ_2 (since it satisfies them with respect to the primed Sigmas), so the conclusion of Dedekind's axiom holds for r.

The case of segments is handled similarly.

b This part of the exercise has itself two parts—the gaps left open in the proof of the fact that Archimedes Axiom follows from Dedekind's Continuity Axiom (pp. 99 and 100 of the text.) For notation, refer to the text. First, we must show that if P and Q are two distinct points in Σ_1 then $PQ \subset \Sigma_1$. We can assume without losing generality that $A * P * Q$. Let R satisfy $P * R * Q$. Then by Prop. 3.3, $A * R * Q$. If R were not covered, then, with $AE_n = n \cdot CD$ (cf. Archimedes Axiom, p. 95), we would have $A * E_n * R$, all $n \geq 1$. Another application of Prop. 3.3 would yield $A * E_n * Q$, $n \geq 1$, so Q would not be covered. But Q is covered (hypothesis), and, consequently, R has to be covered. Therefore, $PQ \subset \Sigma_1$, as desired.

Second, we must fill the gap in Case 2 of the above mentioned proof. Specifically, we are assuming here that O, the point giving the cut of the ray r, is in Σ_2 and we want to show that if P is the point obtained by laying off CD on the ray opposite to Σ_2, then P is in r. Assume, looking for a contradiction, that $P \notin r$. Then, by definition of ray, $P * A * O$ and consequently $AO < OP \cong CD$ so $AO < CD$. On the other hand, since O is not covered then, in particular, $A * E * O$ where $AE = 1 \cdot CD$. This gives $CD < AO$. So under the assumption that $P \notin r$ we get that $CD > AO$ and that $CD < AO$, a contradiction. Therefore P must be in r.

34 In the interpretation described in the statement of the problem, Axiom CI fails for if r is the positive $x-$axis and if $A = (0,0)$ and $B = (1,1)$ then the segment AB cannot be laid on r. Indeed, the length of AB is $\sqrt{2}$ but the point $(\sqrt{2}, 0)$ is not in the model.

The Elementary Continuity Principle fails because, for example, the circle of radius 1 centered at $(0,0)$ and the line $y = x$ do not intersect (at a point in the model, that is) despite the fact that the line has points both inside and outside the circle.

35 I will omit the verifications of the axioms that do hold. For examples of instances of the failures of SAS, the Circular Continuity Principle, the Elementary Continuity Principle, and the SSS, ASA, and SAA criteria, see the sketches in the last page of this handout.

Major Exercises

1 Assume without loss of generality that $r \geq r'$. Then γ and γ' intersect in two distinct points if and only if $d < r + r'$ and $r < d + r'$. The circles are tangent if and only if $d = r + r'$ or $r = d + r'$.