LECTURE 1

PROPERTIES OF FACTORIZATION IN MONOIDS AND CONES

Example 1: natural numbers with multiplication $\mathbb{N} = \{1, 2, 3, \ldots\}$, m **divisor** of n $\quad m|n \quad n = mm'$, $m' \in \mathbb{N}$.

Unique factorziation into prime numbers
Every natural number $n \neq 1$ is a product $n = p_1 p_2 \ldots p_k$ of prime numbers p_i which are uniquely determined by n, (up to renumbering)
$p \in \mathbb{N}$ **prime number** $p \neq 1, p|mn \Rightarrow p|m$ or $p|n$

$$q \in \mathbb{N} \textbf{ atom } q \neq 1, m|q \Rightarrow m = 1 \text{ or } m = q$$

representation by primes/atoms $n = \prod_p \; p^{n(p)}$
different primes p

$$n(p) \geq 0 \text{ and } n(p) > 0 \text{ for only finitely many } p.$$

$(\mathbb{N}, \cdot)$ is a (multiplicative) monoid

$(M, \cdot)$ **monoid** (with cancellation), for $a, b, c \in M$

- $a \cdot b = b \cdot a \in M$
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- $a \cdot b = a \cdot c \Rightarrow b = c$
- neutral element $u \in M$, $a \cdot u = u \cdot a = a \;\; u := 1$.

Example 2: natural vectors with addition
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, vectors $v \in \mathbb{N}_0^d$ with addition
v **summand** of w $v \leq w$ $w = v + v', v' \in \mathbb{N}_0^d$
unique representation by unit vectors.
Every $v \neq 0$ has a unique representation (up to renumbering) $v = v_1 e_1 + \ldots + v_d e_d$ by unit vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, 1 in the i-th component

- e_i is "prime" $e_i \neq 0, e_i \leq v + w \Rightarrow e_i \leq v$ or $e_i \leq w$
- e_i is an atom, $e_i \neq 0, v \leq e_i \Rightarrow v = 0$ or $v = e_i$

  $\mathbb{N}_0^d$ is a cone

  $(C, +)$ **cone**, for $a, b, c \in C$
- $a + b = b + a \in C$
- $a + (b + c) = (a + b) + c$
- $a + b = a + c \Rightarrow b = c$
- neutral element $u \in C$, $a + u = u + a = a$
  $u := 0$

cone = monoid when mult is replaced by addition
geometrical scalar multiplication
λ a scalar, $a \in C \Rightarrow \lambda a \in C$ e.g. $\lambda \in \mathbb{R}_+$

UNFORTUNATELY

MONOIDS DO NOT HAVE UNIQUE FACTOR
CONES DO NOT HAVE UNIQUE FACTOR

MONOIDS

- **Hilbert monoid** $M = \{4n+1 \mid n \in \mathbb{N}_0\}$ with • nonunique fact $441 = 9.49 = 21.21$
  $9, 21, 49$ atoms in M **but** not prime in M
  primes in M? nat prime numbers of form $4n+1 : 5, 13, 17, \ldots$ though infinite **not** enough for representation.
- **polynomial ring** $M = \mathbb{R}[X^2, X^3]$ with nonunique fact $X^6 = X^3 \cdot X^3 = X^2 \cdot X^2 \cdot X^2$
  X^2, X^3 atoms in M **not** prime.
- **algebraic number fields** $M = \mathbb{Z}[\sqrt{-5}]$
  $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$
  different fact into atoms

CONES

polyhedral cone given in $\mathbb{N}_0^d$ by equations.

- $C = \{x \in \mathbb{N}_0^3 \mid 2x_1 + 5x_2 = 3x_3\}$
  nonunique repres
  $(3,3,7) = (1,2,4)+(2,1,3) = (3,0,2)+(0,3,5)$
- $C = \{x \in \mathbb{N}_0^4 \mid x_1 + x_2 = x_3 + x_4\}$
  nonunique repres
  $(1,1,1,1) = (1,0,1,0) + (0,1,0,1) =$
  $(1,0,0,1) + (0,1,1,0)$.
  See problems 1 and 2 for Lecture 1.

FACTORIZATION-GENERAL

$(M, \cdot)$ monoid **reduced** divisors of 1 (unit)$= \{1\}$.

$(C, +)$ cone **pointed** summands of 0 (sero) $= \{0\}$

M **factorial** if **unique factorization** holds:
every $1 \neq x \in M$ is a product of atoms, uniquely determined up to renumbering.
For **integral domains** R mult monoid $(R^{\times}, \cdot)$ factorial
C **simplicial** if **unique representation** holds:
every $0 \neq x \in M$ is a sum of atoms, uniquely determined up to renumbering

Further notions:

- **atom** in $(M, \cdot)$ nonunit $a : a = bc \Rightarrow b$ or c unit
- **atom** in $(C, +)$ nonsero $a : a = b + c \Rightarrow b$ or c sero
- $(M, \cdot)/(C, +)$ is **atomic** if every nonunit/nonsero is a product/sum of atoms. By definition every factorial monoid/simplicial cone is atomic.

SOURCE OF NON-UNIQUENESS ?

Examples indicate prime $\overset{\Rightarrow}{\not\Leftarrow}$ atom
and atoms may decay into atoms

Hilbert monoid	21 atom **but** $(21)^2 = 9.49$
polynomial ring	X^2 atom **but** $(X^2)^3 = X^3 \cdot X^3$
alg number field	3 atom **but** $3^2 = (-2-\sqrt{-5})(-2+\sqrt{-5})$
polyedral cone	$a = (1,2,4)$ atom **but** $3 \cdot a = c + 2d$.

A power/multiple of an atom may decay into atoms

atom $\mathbf{x}$ decays into atoms $\mathbf{y_1, \ldots, y_k}$

if $x^m = y_1 \ldots y_k$ for some m, for monoids
if $mx = y_1 + \ldots + y_k$ for some m, for cones

strong atom atom with trivial decay,
$y_i = x$ or unit/sero.

FROM CONES TO MONOIDS .....

$C = \mathbb{N}_0^3$ leads one to

$$M(C) = \{X_1^{a_1}\, X_2^{a_2}\, X_3^{a_3} \mid (a_1, a_2, a_3) \in C\}$$

monomials in the polynomial ring $\mathbb{R}[X_1, X_2, X_3]$.
$M(C)$ mult monoid by isomorphism
$(a_2, a_2, a_3) \mapsto X_1^{a_1}\, X_2^{a_2}\, X_3^{a_3}$
$(C, +)$ simplicial $\Rightarrow M(C)$ factorial.
More general, any cone $C \subset \mathbb{N}_0^d$ leads to
$M(C) = \{X^a \mid a \in C, a{=}(a_1, \ldots, a_d), X^a = X_1^{a_1} \ldots X_d^{a_d}\}$
monoid of monomials associated to the cone C.

Example $\quad C = \{x \in \mathbb{N}_0^3 \mid 2x_1 + 5x_2 = 3x_3\}$
atoms $a = (1, 2, 4), b = (2, 1, 3), c = (3, 0, 2)$,
$d = (0, 3, 5)$
generate C with relations $3a = c + 2d, 3b{=}2c + d$
$n \in C, n = m_1 a + m_2 b + m_3 c + m_3 d$

$$\begin{array}{cccccc} X^n & = & (X^a)^{m_1} & (X^b)^{m_2} & (X^c)^{m_3} & (X^d)^{m_4} \\ & & U & V & Y & Z \end{array}$$

$U^3 = YZ^2$ by $3a = c + 2d, V^3 = Y^2 Z$ by $3b = 2c + d$

$$\Rightarrow \quad M(C) = \{U^{m_1}\, V^{m_2}\, Y^{m_3}\, Z^{m_4} \mid (m_1, m_2, m_3, m_4) \in \mathbb{N}_0^4, U^3 = YZ^2, V^3 = Y^2 Z\}$$

... TO DOMAINS TO VARIETIES

$C \subseteq \mathbb{N}_0^d$ cone, D domain lead to
semigroup ring $D[C]$ generated by D and $M(C)$
binomial difference of two monomials
variety common zero set of a set of binomials

Examples

binomial $Y^2 - X^3$ **Neil parabola**

binomial $Z^2 - XY$ **quadric cone**

varieties can be very difficult

$$Z^n - X^n - Y^n \text{ in } \mathbb{Q}[X,Y,Z] \text{ variety } = ?$$

Fermat's Last Theorem

Cone & monoid → **Dioid** $(D, \cdot, +)$

$(D, +)$ cone, $(D, \cdot)$ monoid & distributive laws
→ integral domains.

LECTURE 2

KAPLANSKY GEOMETRICALLY: KRULL DOMAINS & POLYHEDRA

Example in Lecture 1: cone generated by finitely many atoms.
Dual description **cone intersection of halfspaces**
Picture • C generated by 4 atoms

- C intersection of 4 halfspaces
- halfspace $P, C \subset P = C - F, F$ facet
- hyperplane $P \cap (-P) = F - F$
- $G = C - C = \mathbb{Z}^3, P \cap (-P)$ 2−dim, $G \diagup_{P \cap (-P)} \cong \mathbb{Z}$

These halfspaces called **prime cones**.

Definition of a prime cone P for C
$(C, +)$ cone, $G = C - C$ group generated by C

- $P \neq G$ **halfspace** P cone with $P \cup (-P) = G$
- $C \subset P = C - F, F$ facet of C
- F **facet** maximal face of $C (\neq C)$
  F **face**, $F \subset C$ cone s.t. $x + y \in F \Rightarrow x, y \in F$.

A prime cone is **discrete** if $P \cup (-P) \diagup_{P \cap (-P)}$ is cyclic group.

A cone is a **Krull cone** if it is the intersection of its prime cones P, which are all discrete, in such a way that for every nonsero x in C only for finitely many P one has that $-x \notin P$.

IRVING KAPLANSKY 1917 -
Commutative rings, revised edition, 1974
characterizes integral domains, which are intersections of certain valuation domains. We give a geometric description in terms of prime cones and recall some fundamental concepts from algebra

- R integral **domain**
- $\wp$ **prime ideal**
- V **valuation domain**, domain s.t. for every $x \in$ quot V^*, $x \in V^*$ or $x^{-1} \in V^*$
- V **discrete** $\text{quot } V^* / \text{units of } V^*$ cyclic group $V^* = V \setminus \{0\}$
- **localisation** R_M, M a monoid in $(R^*, \cdot)$ ring of quotients $\frac{r}{s}, r \in R, s \in M$.

Kaplansky (p. 82): A domain is a **Krull domain** if

- R_M is a discrete valuation domain for every monoid $M = R \setminus \wp, \wp$ minimal prime ideal
- R is the intersection of all R_M, M as above
- every element in R^* lies in only a finite number of minimal prime ideals

Theorem Geometric description of Krull domains
An integral domain R is a Krull domain if and only if $(R^*, \cdot)$ is a Krull cone.
For the proof correspondence

$$\text{valution domains } R_M \longleftrightarrow \text{prime cones}$$

faces of $(\mathbf{R}^*, \cdot)$ F mult closed subset of R^* s.t. $x \cdot y \in F \Rightarrow x \in F, y \in F$, equivalent to F **saturated**, F contains along with an element all its divisors
faces of $(R^*, \cdot) \longleftrightarrow$ saturated mult closed subsets.

Kaplinsky (p. 2) **Theorem 2** R domain, $F \subset R$
F saturated mult closet subset

$$\Leftrightarrow R \setminus F \text{ is a union of prime ideals in } R.$$

Lemma F facet of $(R^*, \cdot)$
$\Leftrightarrow R \setminus F$ is a minimal prime ideal in R.

Proof

"$\Rightarrow$" Kaplansky, Theorem 2: $R \setminus F$ union of $\mathfrak{p}$ prime, $\mathfrak{p} \cap F = \emptyset$ (since F in $(R^*, \cdot)$ mult closed, saturated)
Let $\mathfrak{q}$ prime ideal, $\mathfrak{q} \cap F = \emptyset$. $\mathfrak{q} \subset R \setminus F \Rightarrow F \subset R \setminus \mathfrak{q}$.
$G = R \setminus \mathfrak{q}$ is a face (special case of Kaplansky).
F maximal $\Rightarrow F = G$ (since $\mathfrak{q} \neq \emptyset$)
$\Rightarrow \mathfrak{q} = R \setminus G = R \setminus F$. Thus, $\mathfrak{q} = R \setminus F$ for all $\mathfrak{q}$.
$\mathfrak{q}$ minimal prime ideal: $\mathfrak{q}'$ prime in R,
$\mathfrak{q}' \subset \mathfrak{q} \Rightarrow F = R \setminus \mathfrak{q} \subset R \setminus \mathfrak{q}' = F', F'$ face.
F maximal $\Rightarrow F = F'$ (since $\mathfrak{q}' \neq \emptyset$) $\Rightarrow \mathfrak{q}' = \mathfrak{q}$.

"$\Leftarrow$" Let $\mathfrak{q} = R \setminus F$ minimal prime ideal in R.
$\Rightarrow F$ face (special case of Kaplansky).
Let $F \subset G, G$ face $\neq R^*$.Kaplansky, Theorem 2: $R \setminus G$ union of $\mathfrak{p}$ prime, $\mathfrak{p} \cap G = \emptyset$. Thus, union of $\mathfrak{p} = R \setminus G \subset R \setminus F = \mathfrak{q}$ and, hence,
$\mathfrak{p} \subseteq \mathfrak{q}$ for all $\mathfrak{p}, \mathfrak{p} \cap G = \emptyset$. $\mathfrak{q}$ minimal $\Rightarrow \mathfrak{p} = \mathfrak{q}$ all $\mathfrak{p} \Rightarrow R \setminus G = \mathfrak{q} = R \setminus F \Rightarrow F = G$.
Thus, F maximal. $\square$

Proof of the geometric description of Krull domains
R integral domain
P prime cone in $(R^*, \cdot)$.

$$\Leftrightarrow\ P = {}^{R^*}\!\!/_{F}, \underbrace{F \text{ facet of } R^*}_{\Leftrightarrow_{\text{Lemma}}\ \mathfrak{p} = R \setminus F \text{ minimal prime in } R}, P \cup P^{-1} = R^*$$

$$\Leftrightarrow\ P = {}^{R^*}\!\!/_{C\mathfrak{p}}, \mathfrak{p} \text{ min prime in } R,\ P \cup P^{-1} = R^*.$$

Consider $V = P \cup \{0\} \subset R$. Show that V is an additive group: $x, y \in V \Rightarrow x - y \in V$.

$$x, y \in V \Rightarrow x = \frac{x_1}{x_2}, y = \frac{y_1}{y_2}, x_1, y_1 \in R, x_2, y_2 \notin \mathfrak{p}.$$

$$x - y = \frac{x_1}{x_2} - \frac{y_1}{y_2} = \frac{x_1 y_2 - y_1 x_2}{x_2 y_2} \in {}^{R}\!/_{C\mathfrak{p}} \text{ since } x_2 y_2 \notin \mathfrak{p}.$$

Thus, V is a valuation domain.

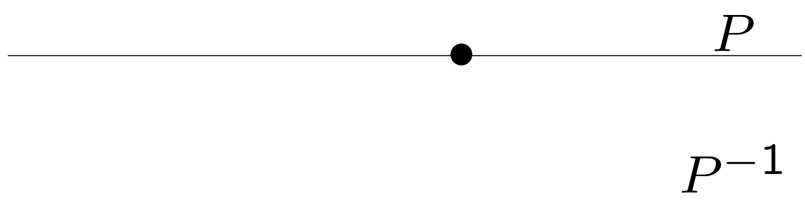
V discrete	$\Leftrightarrow$	$\text{quot} V^* \big/ \text{units } V^*$	cyclic
P discrete	$\Leftrightarrow$	$P \cup P^{-1} \big/ P \cap P^{-1}$	cyclic.

$\square$

Stylized pictures

- **valuation domain/half space** $\quad P = V^*$

$P\backslash F$ minimal prime

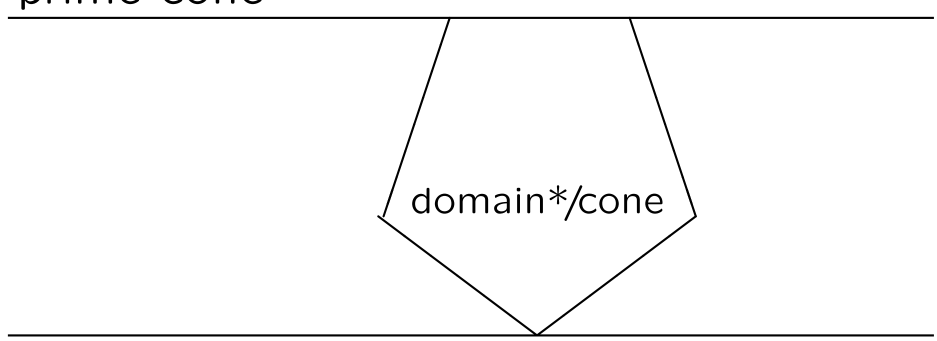


$F = P \cap P^{-1}$
facet of P units of V

$P \cup P^{-1} =$ quot V^*.

- **finite intersections** of valuation domains/half spaces

essential valuation domain/
prime cone

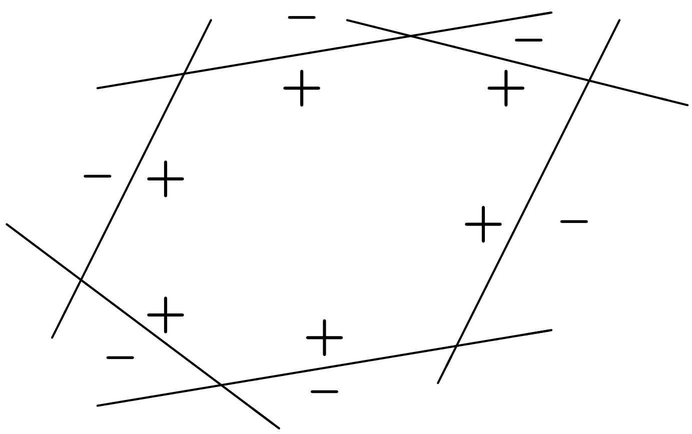


hyperplane/
group of units

not essential valuation domain
not prime/**not** supporting halfspace

geometrically factorial $\leftrightarrow$ simplicial

- **arrangements** of hyperplanes/groups of units partition into chambers
  $\rightarrow$ combinatorial theory of polyhedra/Krull domains



very difficult

Further geometrical descriptions

- **generalized Krull domain/cone**
  valuation domains/prime cones need not be discrete could be **rational** (subgroup of $\mathbb{Q}$) or **real** (subgroup of $\mathbb{R}$) still geometrical description as intersection of prime cones possible

- **Prüfer domain/cone**
  integral domain is a Prüfer domain
  $\mapsto R_M$ valuation domain for all $M = R^* \backslash \mathfrak{p}$,
  $\mathfrak{p}$ prime ideal
  geometrical
  R Prüfer domain $\Leftrightarrow$ valuation domains* are exactly the prime cones
  $\Leftarrow$ intersection of prime cones
  characterization by mult alone possible

- **Dedekind domain**
  Krull domain & Prüfer domain
  $\Rightarrow$ intersection of prime cones
  can not be characterized by multiplication alone

- Krull/Prüfer/Dedekind domain
  $\Rightarrow$ intersection of prime cones

Question Characterize all the domains which are intersections of prime cones.

LECTURE 3

NONUNIQUE BUT FRIENDLY: CALE FACTORIZATION

Source of nonunique factorization in Lecture 1 by examples: atoms can decay into other atoms
interesting feature: this decay may be unique
yields a friendly kind of nonunique factorization called
Cale factorization
Back to earlier examples

- **Hilbert monoid** $M = \{4n + 1 \mid n \in \mathbb{N}\}$ mult
  nonunique factorization $441 = 9.49 = 21.21$
  decay of atom 21 $21^2 = 9.49$
  atoms $9, 49$ do not decay but are strong
  atoms 441 has unique fact into strong atoms

- **polynomial ring** $M = \mathbb{R}[X^2, X^3]$
  nonunique factorization
  $X^6 = X^2 \cdot X^2 \cdot X^2 = X^3 \cdot X^3$
  none of the atoms is strong
  $(X^2)^3 = (X^3)^2$ nonuniqueness **not** friendly.

- **algebraic number fields** $M = \mathbb{Z}[\sqrt{-5}]$ mult nonunique factorization
  $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$
  decay of atoms $3^2 = (-2+\sqrt{-5})(-2-\sqrt{-5})$
  $(1-\sqrt{-5})^2 = 2(-2+\sqrt{-5}), (1-\sqrt{-5})^2 = 2(-2-\sqrt{-5})$
  atoms $2, -2+\sqrt{-5}, -2-\sqrt{-5}$ do not decay
  Cale representation: squares of atoms are unique products of strong atoms
  explains nonuniqueness: just a recording by taking squares
  $6^2 = 2^2(-2+\sqrt{-5})(-2-\sqrt{-5}) = 2(-2+\sqrt{-5})2(-2-\sqrt{-5})$
  3^2 $(1+\sqrt{-5})^2(-1\sqrt{-5})^2$
- **polyhedral cones**
  $C = \{x \in \mathbb{N}_0^3 | 2x_1 + 5x_2 = 3x_3\}$ add nonunique fact
  $(3,3,7) = (1,2,4) + (2,1,3) = (3,0,2) + (0,3,5)$
  decay of atoms $3a = c + 2d, 3b = 2c + d$
  atoms c, d strong atoms do not decay
  Cale representation: multiplied by 3, atoms are unique sums of strong atoms
  explains nonuniqueness: just a recording by taking multiples

$$3 \cdot (3,3,7) = (c+2d) + (2c+d) = 3(c+d)$$

- **However** for $C = \{x \in \mathbb{N}_0^4 \mid x_1 + x_2 = x_3 + x_4\}$ nonunique fact
  $(1,1,1,1) = (1,0,1,0) + (0,1,0,1) = (1,0,0,1) + (0,1,1,0)$
  all four atoms are strong, no decay
  nonuniqueness **not** friendly, no explanation by Cale.

CALE REPRESENTATION-GENERAL

$(M, \cdot)$ monoid, $M^\times$ group of units

Cale representation of $x \in M \backslash M^\times$
with respect to a **base** $Q \subset M \backslash M^\times$

$$x^{m(x)} = u \prod_{q \in Q} q^{x(q)} \qquad \text{unique}$$

in $\mathbb{N}$ minimal unit in $\mathbb{N}$ for finitely many q
tame base to every $q \in Q$ exists $e(q) \in \mathbb{N}$ s.t.

$$e(q) \frac{x(q)}{m(x)} \in \mathbb{N}_0 \quad \text{for } \textbf{all} \quad x \in M \backslash M^\times$$

M	**Cale monoid**	Cale representation & tame base
C	**Cale cone**	$(C, +)$ is a Cale monoid
R	**Cale domain**	$(R^*, \cdot)$ is a Cale monoid

Question What is the general relationship between unique fact & Cale repres?

Theorem A monoid is factorial if and only if

- it is a Cale monoid
- it is atomic with all atoms being strong.

Proof without loss, M reduced

$\Rightarrow$: Q set of atoms, $Q \neq \emptyset$.
$x^{m(x)} = \prod q^{x(q)}$, for $x \neq 1, m(x) = 1, x(q) > 0$
finite $q \in Q$, unique repr. $x(q)/m(x) \in \mathbb{N}_0$ all $x \neq 1$.
Atom x decays $\Rightarrow x^m = y_1 \dots y_k \Rightarrow y_i = x$ all i
$\Rightarrow x$ strong atom.

$\Rightarrow$: Choose base Q of atoms as follows

$$\begin{aligned}
q_0 \in Q \quad &\Rightarrow \quad q_0 = a_1 \cdot a_2, a_1, a_2 \text{ atoms.} \\
a_i^{m_i} \quad &= \quad \prod q^{\alpha_i(q)} \Rightarrow \\
q_0^{m_1 m_2} \quad &= \quad \prod q^{\alpha_1(q) m_2} \prod q^{\alpha_2(q) m_1} \\
&\quad\quad = \prod q^{\alpha_1(q) m_2 + \alpha_2(q) m_1} \\
\Rightarrow \quad & \quad\quad \alpha_1(q) = \alpha_2(q) = 0 \text{ for } q \neq q_0 \\
&\Rightarrow \quad a_1^{m_1} = q_0^{\alpha_1(q_0)}
\end{aligned}$$

Exchanging $q_0 \in Q$ for a_1,

$$\begin{aligned}
x^{m(x)} \quad &= \quad \prod_{q \neq q_0} q^{x(q)} \cdot q_0^{x(q_0)} \Rightarrow \\
x^{m(x)\alpha_1(q_0)} \quad &= \quad \prod_{q \neq q_0} q^{x(q)\alpha_1(q)} \cdot a_1^{x(q_0)} \text{ still unique}
\end{aligned}$$

x strong atom $\Rightarrow x^{m(x)} = \prod q^{x(q)}$ yields $x = q' \in Q$.
Thus, Q set of all atoms. M atomic $\Rightarrow$
$x^{m(x)} = \prod q^{x(q)}, x \neq 1, m(x) = 1$.
Unique Cale repr $\Rightarrow M$ factorial. $\square$

CRITERIA FOR CALE REPRESENTATION

- **Numerical semigroups** $C = \mathbb{N}_0 g_1 + \ldots + \mathbb{N}_0 g_\ell$ add $g_i \in \mathbb{N}$, pairwise relative prime, $g_1 < g_2 \ldots < g_\ell$, $Q = \{g_1\}$ is a Cale base, e.g. $C = \mathbb{N}_0 \cdot 2 + \mathbb{N}_0 \cdot 3$, $\rightarrow$ Problems
- **Diophantine cones**
  $C = \{x \in \mathbb{N}_0^{n+1} \mid a_1 x_1 + \ldots + a_n x_n = b x_{n+1}\}$ add is a Cale cone for $a_i \in \mathbb{N}_0, b \in \mathbb{N}$.
  Strong atoms are given by

$$\begin{aligned} q_i &= \frac{b}{gcd(a_i, b)} e_i + \frac{a_i}{gcd(a_i, b)} e_n, 1 \leq i \leq n \\ Q &= \{q_1, \ldots, q_n\} \text{ is a tame Cale base, e.g.} \\ C &= \{x \in \mathbb{N}_0^3 \mid 2x_1 + 5x_2 = 3x_3\} \rightarrow \text{ Problems} \end{aligned}$$

  (Chapmann, Krause 2004)
- **Algebraic number fields**
  R principal order of an algebraic number field is a Cale domain with a Cale base consisting of strong atoms (Chapmann, Halter-Koch, Krause 2002), e.g. $R = \mathbb{Z}[\sqrt{-5}]$ in the number field $\mathbb{Q}(\sqrt{-5})$.
- **Diophantine cones** above are Krull monoids with torsion class
  group $C = \{x \in \mathbb{N}_0^4 \mid x_1 + x_2 = x_3 + x_4\}$ no torsion class group. Principal order of alg number$f$ is a Krull domain with finite class group other orders **not** necessarily Krull domains.

- **Krull monoids**
  A monoid M is a Krull monoid with torsion class group if and only if
  - M is a Cale monoid
  - M is root closed (i.e., $(\frac{a}{b})^k \in M$ for some $k \Rightarrow \frac{a}{b} \in M$, where $a, b \in M$).

  By the geometric characterization of Krull domains in Lecture 2:

- **Krull domains**
  A domain D is a Krull domain with torsion class group if and only if
  - D is a Cale domain
  - D is root closed (for mult).

- **Semigroup rings**
  D domain, integrally closed (roots in quot D of monic polynomials with coeff in D are already in D)
  S cone, pointed, root closed
  $D[S]$ Cale domain $\Leftrightarrow$ D Cale domain & S Cale cone

- **Gilmer-Parker Theorem**
  $D[S]$ factorial $\Leftrightarrow$ D factorial &S simplicial

  Cf. Chapman, Halter-Koch, K 2002;
  K 2004; Chapman, K 2006.

LECTURE 4

TORIC IDEALS, CALE VARIETIES & TROPICAL GEOMETRY

The theory of toric varieties plays an important role at the crossroads of geometry, algebra and combinatorics.
Bernd Sturmfels, Gröbner bases & convex polytopes, p. 127.

Back to cones, most simple & nontrivial case

$C = \{x \in \mathbb{N}_0^3 \mid x_1 + x_2 = 2x_3\}$, Problem 1(i) Lect 1

atoms $q_1 = (2, 0, 1), q_2 = (0, 2, 1), a = (1, 1, 1)$
Cale representation $2a = q_1 + q_2 \to$ cf. Lect 1

- binomial $Z^2 - XY$
- variety = zero set quadric cone

General C Cale cone $\to$ semigroup ring $D[C] \to$
$\to$ toric ideal generated by binomials $\to$ Cale variety/ zero set.

D integral domain
C cone finitely generated by $g_1, \dots, g_\ell$
Rédei map $\Phi : \mathbb{N}_0^\ell \to C, \Phi(a) = \sum_{i=1}^{\ell} a_i g_i$

$$\Phi^* : D[\mathbb{N}_0^\ell] \to D[C], \sum r_a X^a \to \sum r_a X^{\Phi(a)}$$

surjective, $I =$ Ker Φ^* **toric ideal of** $\mathbf{C}$ for $g_1, \dots, g_\ell$
$V(I) =$ zero set of polynomials in I,
called **generalized affine toric variety** (Sturmfels)
coordinate ring $D[C] \cong^{D[\mathbb{N}_0^\ell]} \diagup_{I/(C)}$
binomials $X^a - X^b, a, b \in \mathbb{N}_0^\ell$, $\min\{a_i, b_i\} = 0$
generate I, Gilmer, Sturmfels
Cale variety $V_C = V(I), I$ toric ideal of a Cale cone C
Cale representation $Q = \{g_1, \dots, g_k\}$ Cale base

$$m_j g_j \;=\; \sum_{i=1}^{k} c_{ij} g_i, \quad k+1 \le j \le \ell$$

$$\alpha_j \;=\; m_j e_j, \quad \beta_j = \sum_{i=1}^{k} c_{ij} e_i \text{ in } \mathbb{N}_0^\ell$$

base binomials $X^{\alpha_j} - X^{\beta_j}, \quad k+1 \le j \le \ell.$

CALE VARIETIES Chapman/K 2005

C Cale cone, finitely generated, D integral domain

Theorem 1 The Cale variety V_C is the zero set of all base binomials, that is it is given by equations

$$X_j^{m_j} = X_1^{c_{1j}} X_2^{c_{2j}} \dots X_k^{c_{kj}}, \quad k+1 \le j \le \ell.$$

Geometric-algebraic correspondence

$V \subset D^n$ **homogeneous**	C **half-factorial**
$\lambda \in D^*, y \in V \Rightarrow \lambda y \in V$	all factorizations of a fixed $0 \neq x \in C$ have same number of atoms

Theorem 2 Equivalent for D with char $D \neq 2$:

- V_C homogeneous
- C half-facctorial
- in the Cale repres $m_j = \sum_{i=1}^{k} c_{ij}, k+1 \le j \le \ell$
- all binomials in $I(C)$ are homogeneous
- all base binomials are homogeneous.

For any finitely generated Krull cone (Lecture 2)

Theorem 3 Equivalent for D a field

- V_C Cale variety
- coordinate ring $D[C]$ is a Cale domain
- C has torsion class group

Remarks

- in Theorem 3, V_C is automatically normal ($D[C]$ integrally closed, C normal/root closed) $V(I)$ in general **not** normal
- in Theorem 3, C turns out to be a Diophantine cone because of
  C finitely generated Krull cone with torsion class group
  $\Leftrightarrow C$ Dioph cone given by finitely many equations
  $a_1x_1 + \ldots + a_{n-1}x_{n-1} = bx_n$
  $$a_i \in \mathbb{N}_0, \in \mathbb{N}.$$

Question Conditions for a finitely generated Cale cone to be half-factorial? Difficult even for Dioph cone given by just 1 equation (Chapman, Oeljeklaus, K 2000).

EXTENDED EXAMPLE

$$C = \{x \in \mathbb{N}_0^n \mid x_1 + x_2 + \ldots + x_{n-1} = bx_n\}, \ \ b \in \mathbb{N}$$

see also earlier examples and problems
Diophantine cone, cancell, torsion-free, seros = $\{0\}$.
C Cale cone, $q_i = be_i + e_n, 1 \leq i \leq n-1$, Cale base
Cale representation for $x = (x_1, \ldots, x_n) \in C$:

$$\begin{aligned} bx &= bx_1e_1 + \ldots + bx_ne_n = \\ & x_1q_1 + \ldots + x_{n-1}q_{n-1} + bx_ne_n - \\ & e_n(x_1 + \ldots + x_{n-1}) \\ \Rightarrow & \quad bx = x_1q_1 + \ldots + x_{n-1}q_{n-1} \end{aligned}$$

Obviously, C is finitely generated.

Questions

- Find a generating set, i.e. all atoms of C.
- Find all base binomials.
- How does the Cale variety for C look like?
- What properties has the Cale variety?
  In particular, is it homogeneous?

Set of generating atoms for C

$x \in C$ atom $\Rightarrow x_n = 1$; otherwise, x_i can be reduced $x_1 + x_2 + \ldots + x_{n-1} = b, x = (\bar{x}, 1), \bar{x}$ a partition of b. More precisely $p = (p_1, \ldots, p_r)$ partition of b if $p_i \in \mathbb{N}$ $p_1 \leq \ldots \leq p_r < b$ and $p_1 + \ldots + p_r = b$. Identify atoms not in Cale base $Q = \{q_1, \ldots, q_k\}$ by

$$x \text{ atom } \Leftrightarrow \bar{x} \text{ is a permutation of a partition of } b \text{ (only } p_i \neq p_j \text{ permuted)}$$

base binomials are given by atoms not in Qg_j, $k+1 \leq j \leq \ell, p(j)\bar{g}_j$ permutet pastition

$$\begin{aligned} X_j^b &- X^{p(j)} \text{ in } D[X_1, \ldots, X_k, \ldots, X_\ell] \\ X^P &= X_1^{P1} \ldots X_r^{P_r} \end{aligned}$$

Cale variety V_C given by equations $X_j^b = X^{p(j)}, k+1 \leq j \leq \ell$. Corresponds to the $b-$th **Veronese anbedding** of projective space P^{n-2} (Sturmfels, Gröbner Bases ..., Chapter 14).

To determine the Cale variety we dont need to know all generators of the toric ideal

homogeneity/half-factoriality

Cale representation $bx = x_1 q_1 + \ldots + x_{n-1} q_{n-1}$ and $b = x_1 + \ldots x_{n-1}$ for atoms imply that C is halffactorial and V_C homogeneous.

base polynomials $X_j^b - X^{p(j)}$ homogeneous.

Particular cases

case $\mathbf{b = 1}$	Cale base consists of all atoms
	$\Rightarrow C$ is factorial, $C \cong N_0^{n-1}$
Exercise	Show that, conversely, C-factorial
	implies that $b = 1$.

case $\mathbf{b = 2}$ with permutation

$$p = e_i + e_h, \quad 1 \leq i \neq h \leq n-1.$$

base binomials $X2_j - X_i X_h$.
Cale variety intersection of $\binom{n-1}{2}$ cone quadrics

Exercise Make a picture of the possible intersection of two cone quadrics

$$\sharp \text{ of all atoms } = (n-1) + \binom{n-1}{2} = \binom{n}{2}.$$

The base binomials do **not** generate the toric ideal $I(C =$: A minimal set of generators of $I(C)$ contains in addition binomials of type $X_i X_j - X_h X_k$.

$\mathbf{b=3}$ $\quad 1+1+1=3, 1+2=3$ typical partitions with permutation $p=e_i+e_h+e_k, p=e_i+e_h$
i,j,h,k different
base binomials $\quad X_j^3-X_iX_hX_k, X_j^3-X_iX_h^2$
the correspondong varieties known as **twisted cubic curves**
$\mathbf{n=3}$ $\quad X_3^3-X_1X_2^2, X_4^3-X_1^2X_2$

Exercise Compute picturs of twisted cubic curves and their intersection

See earlier example $C=\{x\in\mathbb{N}_0^3 \mid x_1+x_2=3x_3\}$ which is isomorphic to $C=\{x\in\mathbb{N}_0^3 \mid 2x_1+5x_2=3x_3\}$ (see Problem 1 in Lecture 1, Problem 1 in Lecture 2).

The number of partitions of a number b increases fast with the size of b

$\rightarrow$ number of atoms increases fast
$\rightarrow$ number of base binomials increases fast
$\rightarrow$ Cale variety difficult kind of intersection.