OVERLOOKED FACTORIZATION DOMAINS

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Amanda Matson

North Dakota State University Graduate Student

Unique Factorization Domains

One of the most prominent types of factorization domains is a Unique Factorization Domain (UFD); that is, a ring in which every nonzero nonunit can be written as a finite product of prime elements. UFD's need be atomic by construction. It is of interest to study what happens when atomicity is removed. There are two main definitions that describe nonatomic UFD's.

Defn: An integral domain, R, is an Atoms-Prime domain (AP-domain) if every nonzero nonunit irrreducible of R is prime.

Defn: An integral domain, R, is an Unrestricted Unique Factorization Domain (U-UFD) if every nonzero nonunit of R that can be factored into irreducibles does so uniquely.

Certainly if we add atomic to either of the above definitions, we would return to a UFD. Consider for a moment an atomic AP-domain, R. Since R is atomic, every nonzero nonunit of R can be written as a finite product of irreducibles. Since R is an AP-domain, all of those irreducibles are prime. It follows directly that every nonzero nonunit of R can be written as a finite product of primes and R is a UFD. Next we consider what happens when R is an atomic U-UFD. Since R is atomic, every nonzero nonunit can be written as a finite product of irreducibles. Since R is a U-UFD, each of those elements factors uniquely forcing R to be a UFD.

The next main question is to determine whether AP-domains and U-UFD's are equivalent concepts or not.

Thm: Any AP-domain is a U-UFD.
Pf: Let R be an AP-domain. Let r be an element of R which factors into a finite product of irreducibles. Since R is an AP-domain, each of those irreducibles is prime so r factors into a finite product of prime elements. Since r factors into a finite product of prime elements, the factorization of r is unique and, as this happens for every r in R that factors into a finite product of irreducibles, R is a U-UFD. $\square$

Ex: For an example of an AP-domain, and hence a U-UFD, consider the ring $\mathbb{Z}+x\mathbb{Q}[[x]]$. This ring is not atomic; however, every irreducible is prime.

Thm: There exist U-UFD's which are not AP-domains.
Pf: The construction of such a domain is in the paper "AP Domains and Unique Factorization" by J. Coykendall and M. Zafrullah which can be found in Journal of Pure and Applied Algebra, 2004. We will give a brief sketch of the construction here. The reader is encouraged to consult the aforementioned paper for more details.

- Let R be an integral domain with at least one nonprime irreducible element, π. Set R_0 to be R.
- Collect the non π irreducibles of R_0 into the set $\{\alpha^{(0)}_{i^{(0)}}\}_{i^{(0)}\in\Lambda^{(0)}}$.

- Build R_1 as $R_0[x^{(0)}_{i^{(0)}}]\left[\frac{\alpha^{(0)}_{i^{(0)}}}{x^{(0)}_{i^{(0)}}}\right]_{i^{(0)}\in\Lambda^{(0)}}$.
- Notice that, at this point, there may be other irreducibles in R_1 that we do not want there. To solve this problem, repeat this process indefinitely.
- Collect the non π irreducibles of R_n into the set $\{\alpha^{(n)}_{i^{(n)}}\}_{i^{(n)}\in\Lambda^{(n)}}$.
- Build R_{n+1} as $R_0[x^{(n)}_{i^{(n)}}]\left[\frac{\alpha^{(n)}_{i^{(n)}}}{x^{(n)}_{i^{(n)}}}\right]_{i^{(n)}\in\Lambda^{(n)}}$.
- Observe now that we have: $R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq \cdots$.
- Take $T = \cup_{i=0}^{\infty} R_i$.
- Show that π is irreducible in T and that π is the unique irreducible of T.
- Show that T is nonatomic.
- Show that π is not prime in T.

The domain T is a U-UFD which is not an AP-domain. $\square$

This type of construction was also used by B. Mammenga in her doctoral dissertation to show the existence of an integral domain with factorization type isomorphic to any given monoid which is cancellative, reduced and torsion free.

The questions that can be asked about U-UFDs and AP-domains have not been exhausted as of yet. The reader is encouraged as always to pursue these further.

Finite Factorization Domains and Bounded Factorization Domains

We will next consider what happens when we remove atomicity from other factorization domains. We will consider first Finite Factorization Domains (FFD's) and then Bounded Factorization Domains (BFD's). Before we begin, we will recall some useful definitions and see some examples.

For the time being, we refer to the set of finite factorizations of an element, a, into irreducibles up to units and order as $Z(a)$. We refer to the sets of lengths of those finite factorizations for a given element as $L(a)$. For a quick example of what these refer to, consider the set of integers, $\mathbb{Z}$, and the element, 12. $Z(12)$ is simply $\{2 * 2 * 3\}$ and $L(12)$

is $\{3\}$ as there is only one factorization of 12 into irreducibles up to units and order and it consists of 3 irrreducibles.

Defn: An integral domain, R, is an FFD if given any nonzero nonunit in R, then there is a finite number of ways to factor it into atoms. Using the terminology described above, for any nonzero nonunit a of R, $|Z(a)|$ is finite.

Ex: For various examples of FFD's the reader is referred to the paper "Finite Factorization Domains" by D.D. Anderson and B. Mullins which appeared in the Proc. Amer. Math. Soc. 1996. For one such example from the paper, we consider, R, an integral domain with $R/(a)$ finite for each nonzero nonunit a in R. Such a ring is an FFD.

Defn: An integral domain, R, is a BFD if given any nonzero nonunit of R, then there is a bound to the length of its factorizations. Using the terminology described above, for any nonzero nonunit a in R, $|L(a)|$ is finite. This is the case primarily because $L(a)$ consists of nonnegative integers.

Ex: For good examples of BFD's, recall that any Noetherian domain is a BFD. One such ring is $\mathbb{Z}[x, y]$.

Defn: A ring R is atomic if every nonzero nonunit of R can be written as a finite product of irreducibles. Using the terminology described above, R is atomic if $Z(a)$ is nonempty for every nonzero nonunit a of R.

Now that we have an understanding of FFD's and BFD's, we again question what we are left with when we remove atomicity from each of them in turn. We begin with a possible definition of an unrestricted FFD.

Defn: Let R be an integral domain. If for every nonzero nonunit, a, of R, $Z(a)$ is empty or $|Z(a)|$ is finite, then we say that R is an unrestricted FFD (U-FFD).

The most striking question that we come across here is regarding the difference between an idf-domain and a U-FFD. Recall that in an idf-domain, every nonzero nonunit which can be written as a finite product of irreducibles has only finitely many irreducible divisors. It is my belief that these are different concepts and that if we looked far enough, we would find examples forbidding both implications. We will not discuss this further here, but as us usual, the reader is encouraged to pursue this concept.

Defn: Let R be an integral domain. If it holds that whenever r is a nonzero nonunit of R such that $Z(r)$ is nonempty then $L(r)$ is finite, then we call R an unrestricted BFD (U-BFD).

It would be of interest to see an example of a U-BFD which is not a BFD. It would be nice to see if the same diagram from before holds without the assumption of atomic. It would be nice to see how the weaker nonatomic versions will effect the diagram.