Monoids and Combinatorics I

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The Art of Factorization in Multiplicative Structures
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Numerical monoids

Take any $n_1, \ldots, n_t \in \mathbb{N}$ and consider the *numerical monoid*

$$\langle n_1, \ldots, n_t \rangle := \{a_1 n_1 + \cdots + a_t n_t \mid a_i \in \mathbb{N}_0\}$$

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Example

$\langle 4, 6 \rangle = \{0, 4, 6, 8, 10, 12, 16, 14, 18, \ldots\}$

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$\langle 3, 7, 10 \rangle = \{0, 3, 6, 7, 9, 10, 12, 13, 14, \ldots\}$

Minimal Presentation

$\langle 4, 6\rangle = \{0, 4, 6, 8, 10, 12, 16, 14, 18, \ldots\}$ never has an odd element because $\gcd(4, 6) = 2$. Naturally:
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Definition

A numerical monoid $\langle n_1,\ldots,n_t\rangle$ is *minimally presented* if $\gcd(n_1,\ldots,n_t) = 1$ and $n_1,\ldots,n_t$ is a minimal set of generators.

Frobenius Number

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Finding the Frobenius number for an arbitrary numerical monoid (varying t) is NP-hard. [Ramírez-Alfonsín]

Factoring

Consider $\langle 3, 7, 8 \rangle = \{0, 3, 6, 7, 8, 9, \ldots\}$. Factorizations of 56?

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$$\begin{aligned}
56 &= 0\cdot 3 + 0\cdot 7 + 7\cdot 8\\
&= 0\cdot 3 + 8\cdot 7 + 0\cdot 8\\
&= 8\cdot 3 + 0\cdot 7 + 4\cdot 8\\
&= 16\cdot 3 + 0\cdot 7 + 1\cdot 8\\
&= 7\cdot 3 + 5\cdot 7 + 0\cdot 8\\
&= 14\cdot 3 + 2\cdot 7 + 0\cdot 8\\
&= 3\cdot 3 + 1\cdot 7 + 5\cdot 8\\
&= 6\cdot 3 + 2\cdot 7 + 3\cdot 8\\
&= 9\cdot 3 + 3\cdot 7 + 1\cdot 8\\
&= 11\cdot 3 + 1\cdot 7 + 2\cdot 8\\
&= 4\cdot 3 + 4\cdot 7 + 2\cdot 8
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&= 0 \cdot 3 + 8 \cdot 7 + 0 \cdot 8 && L = 8 \\
&= 8 \cdot 3 + 0 \cdot 7 + 4 \cdot 8 && L = 12 \\
&= 16 \cdot 3 + 0 \cdot 7 + 1 \cdot 8 && L = 17 \\
&= 7 \cdot 3 + 5 \cdot 7 + 0 \cdot 8 && L = 12 \\
&= 14 \cdot 3 + 2 \cdot 7 + 0 \cdot 8 && L = 16 \\
&= 3 \cdot 3 + 1 \cdot 7 + 5 \cdot 8 && L = 9 \\
&= 6 \cdot 3 + 2 \cdot 7 + 3 \cdot 8 && L = 11 \\
&= 9 \cdot 3 + 3 \cdot 7 + 1 \cdot 8 && L = 13 \\
&= 11 \cdot 3 + 1 \cdot 7 + 2 \cdot 8 && L = 14 \\
&= 4 \cdot 3 + 4 \cdot 7 + 2 \cdot 8 && L = 10
\end{aligned}
$$

Factoring

Length set: $\mathscr{L}(56) = \{7, 8, 9, 10, 11, 12, 13, 14, 16, 17\}$

Elasticity: $\rho(56) = \frac{\max \mathscr{L}(56)}{\min \mathscr{L}(56)} = \frac{17}{7}$

Delta Set: $\Delta(56) = \{1, 2\}$

Elasticity

The elasticity of a monoid M is defined to be

$$\rho(M) := \sup_{x \in M} \rho(x) = \sup_{x \in M} \frac{\max \mathscr{L}(x)}{\min \mathscr{L}(x)}$$

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What is $\rho(\langle n_1, \ldots, n_t \rangle)$?

$\rho(\langle n_1, \ldots, n_t \rangle) = \frac{n_t}{n_1}$

Elasticity

For any factorization:
$x = a_1 n_1 + \ldots + a_t n_t \leq (a_1 + \ldots a_t) n_t$
so $\min \mathscr{L}(x) \geq \frac{x}{n_t}$.

Similarly, $x = a_1 n_1 + \ldots + a_t n_t \geq (a_1 + \ldots a_t) n_1$, so $\max \mathscr{L}(x) \leq \frac{x}{n_1}$.

Therefore $\rho(x) \leq \frac{\frac{x}{n_1}}{\frac{x}{n_t}} = \frac{n_t}{n_1}$ for all $x \in M$.

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But $n_1 n_t$ can be factored as n_t copies of n_1 and as n_1 copies of n_t,
so $\rho(n_1 n_t) = \frac{n_t}{n_1}$.

$\rho(\langle n_1, \ldots, n_t \rangle) = \frac{n_t}{n_1}$.

Delta Sets

If $\mathscr{L}(x) = \{l_1, \ldots, l_r\}$ is listed in increasing order, then

$$\Delta(x) = \{l_{i+1} - l_i \mid 1 \leq i < r\}$$
$$\Delta(M) = \bigcup_{x \in M} \Delta(x)$$

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Recall: in $\langle 3, 7, 8 \rangle$, we had $\Delta(56) = \{1, 2\}$. What is $\Delta(\langle 3, 7, 8 \rangle)$?

Delta Sets

Theorem (Bowles, Chapman, Kaplan, Reiser 2006)

If $M = \langle n_1, n_1 + k, \ldots, n_1 + rk \rangle$, *then* $\Delta(M) = \{k\}$.

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Corollary

$\Delta(\langle n_1, n_2 \rangle) = \{n_2 - n_1\}$

3-generated Numerical Monoids

Fact (Geroldinger 1991)

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Example

For all n, we have

$$\Delta(\langle n, n+1, n^2 - n - 1\rangle) = \{1, 2, \ldots, n-2, 2n-5\}$$

Delta Set Questions

1. Determine $\Delta(\langle n_1, n_2, n_3 \rangle)$.

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2. Is there a numerical monoid M such that $|\Delta(M)| \geq 2$ and $1 \in \Delta(M)$ but $2 \notin \Delta(M)$?

Periodicity

Observations indicate that as x varies through the numerical monoid to ∞, the function $\Delta(x)$ is eventually periodic. Supported by Hassler's work on C-monoids.

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3. Find an explicit N after which point $\Delta(x)$ is periodic.

4. The period always appeared to be a multiple of n_1. Is this always true? What's the maximum multiple of n_1 that is possible?

Definition

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If M is a (reduced) atomic monoid, then $\mathscr{A}(M)$ is the set of all atoms of M. The *monoid of factorizations* is $Z(M) = \mathscr{F}(\mathscr{A}(M))$, the free abelian monoid on $\mathscr{A}(M)$.

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$$\pi(z) = \prod_{u \in \mathscr{A}(M)} u^{v_u(z)}$$

For each $a \in M$, $Z(a) := \pi^{-1}(a) \subset Z(M)$ is the set of all factorizations of a.

Distance Between Factorizations

$Z(M)$ has a natural distance function $d : Z(M) \times Z(M) \to \mathbb{N}_0$ given by:

$$d(y, z) = \max\{|x^{-1}y|, |x^{-1}z|\},$$

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Show that

$$d(y,z) = \frac{1}{2}\left[\sum_{u\in\mathscr{A}(M)} |v_u(y) - v_u(z)| + \left|\sum_{u\in\mathscr{A}(M)} v_u(y) - v_u(z)\right|\right]$$

Distance Properties

We have the following properties:

1. $d : Z(M) \times Z(M) \to \mathbb{N}_0$ is a metric.
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3. $d(xy, xz) = d(y, z)$.
4. $d(y^k, z^k) = kd(y, z)$.
5. $d(yy', zz') \leq d(y, z) + d(y', z')$

Catenary Degree

Definition

The *catenary degree* $c(a)$ of an element $a \in M$ is the least integer c such that between any $z, z' \in Z(a)$ there exists $z_0 = z, z_1, \ldots, z_k = z'$ such that $d(z_i, z_{i+1}) \leq c$ for all i.

Exercise

Question

If M is not factorial, then for every $k \in \mathbb{N}$, there is an $a \in M$ such that $|Z(a)| \geq k+1$ and there are $z, z' \in Z(a)$ such that $d(z, z') \geq 2k$.

Monoid of Trades

Definition

The *monoid of trades* $T(M)$ is:

$$T(M) := \{(1,1)\} \cup \bigcup_{a \in M} Z(a) \times Z(a)$$

This is a submonoid of $Z(M) \times Z(M)$ and so $d(\cdot,\cdot)$ restricts naturally to $T(M)$.

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This is a submonoid of $Z(M) \times Z(M)$ and so $d(\cdot, \cdot)$ restricts naturally to $T(M)$. Again, we wish to recover the monoid elements, so we equip $T(M)$ with a homomorphism $\pi : T(M) \to M$, defined as $\pi(z, y) = \pi(z)$.

Trading Numerical Monoids

For a finitely-generated monoid M (for example, if M is a numerical monoid $\langle n_1, \ldots, n_t \rangle$), the monoid of trades $T(M)$ is also finitely generated. Particularly $\mathscr{A}(T(M))$ is finite.

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Other Undergraduate Projects

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2. [Hine, Paixão 2006] Studied the functions:

$$\begin{aligned} \mathscr{V}(n) &:= \bigcup \{\mathscr{L}(x) \mid n \in \mathscr{L}(x)\} \\ \Phi(n) &:= |\mathscr{V}(n)| \\ \Delta_{\mathscr{V}}(n) & \\ \Delta_{\mathscr{V}}(M) &:= \bigcup_{n \in \mathbb{N}} \Delta_{\mathscr{V}}(n) \end{aligned}$$

and found that for monoids M where $\Phi(n)$ is finite for all n:

$$\frac{\rho(M)^2 - 1}{\rho(M) \max \Delta_{\mathscr{V}}(M)} \leq \liminf_{n\to\infty} \frac{\Phi(n)}{n} \leq \limsup_{n\to\infty} \frac{\Phi(n)}{n} \leq \frac{\rho(M)^2 - 1}{\rho(M) \min \Delta_{\mathscr{V}}(M)}$$