

Monoids and Combinatorics II

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The Art of Factorization in Multiplicative Structures
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Elasticity Definitions

Definition

For $x \in M$, where M a bounded factorization monoid, the *elasticity* of x is $\rho(x) = \frac{\max \mathcal{L}(x)}{\min \mathcal{L}(x)}$. The *elasticity* of M is $\rho(M) = \sup_{x \in M} \rho(x)$.

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M is *fully elastic* if for every $q \in \mathbb{Q} \cap [1, \rho(M))$, there is an $x \in M$ with $\rho(x) = q$. We denote $\mathcal{R}(M) = \{\rho(x) \mid x \in M\}$.

Finitely-Generated Monoids

Theorem

If M is a finitely-generated monoid, then M has finite elasticity and it is accepted.

Example

Consider $M = \mathcal{F}(a, b)/(a^2 = b^3)$. Every element can be written uniquely as $a^r b^s$ for some $0 \leq s \leq 2$. Given such an element, what are its factorizations?

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This gives a length set of:

$$\mathcal{L}(a^r b^s) = \{r + s, r + s + 1, r + s + 2, \dots, r + s + \lfloor \frac{r}{2} \rfloor\}$$

so

$$\rho(a^r b^s) = 1 + \frac{\lfloor \frac{r}{2} \rfloor}{r + s}$$

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If $s = r$ then this monoid is half-factorial, so trivially fully elastic.

Monoid Incognito!

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When $r \not\mid s$, then $\mathcal{F}(a, b)/(a^r = b^s) \cong \langle r, s \rangle$.

Theorem (Chapman, Holden, Moore)

*Any nontrivial numerical monoid is **not** fully elastic.*

Numerical Monoids as Quotients

Every numerical monoid is a quotient of the form:

$$\mathcal{F}(a_1, \dots, a_n) / (\sigma_1, \dots, \sigma_r)$$

for some relations σ_i on the generators.

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Example

$\langle 3, 7, 8 \rangle =$

$\mathcal{F}(a_1, a_2, a_3) / (a_1^7 = a_2^3, a_1^8 = a_3^3, a_2^8 = a_3^7, a_1^3 a_2 = a_3^2, a_1^2 a_3 = a_2^2, a_2 a_3 = 5a_1).$

All these relations are necessary.

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Proof: If a_i does not appear in any σ_j , then it is involved in no unusual factorizations. In other words, a_i is prime. Now use:

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Theorem (B., et al.)

If M has a prime and for every $x \in M$ there is an m such that $\rho(x^m) = \lim_{n \rightarrow \infty} \rho(x^n)$, then M is fully elastic.

This second property is known in the literature as M is *taut*.

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Block monoids are $\mathcal{F}(G^x)$ quotiented out by relations. These relations are *completely dependent*.