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Problems for Lectures 1 & 2 for the MAA PREP Workshop

The Art of Factorization in Multiplicative Structures

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A **cone** is an additively written commutative semigroup C with neutral element 0 and cancellation law ($a+b=a+c$ implies $b=c$). For the following problems we assume that C is **pointed**, $C\cap(-C)=\{x\in C \mid -x\in C\}=\{0\}$.
An element $0\neq a\in C$ is called an **atom** in C if $a=b+c$ and $b,c\in C$ implies that $b=0$ or $c=0$.

Lecture 1: Properties of factorization in monoids and cones

1. Find all atoms for the following cones C.

   (i) $C = \{x \in \mathbb{N}_0^3 \mid x_1 + x_2 = 2x_3\}, \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

   (ii) $C = \{x \in \mathbb{N}_0^4 \mid x_1 + x_2 = x_3 + x_4\}$

   (iii) Compare the cone in (i) with the cone

   $$C = \{x \in \mathbb{N}_0^3 \mid x_1 + x_2 = 3x_3\}.$$

   (iv) Can you find relations/equations among the atoms in a cone as above?

   (v) Draw a picture of one of the above cones. (By hand or by using a computer.)

2. Let for elements a, b of a cone $a \leq b$ iff $b = a + c$ for some $c \in C$. An element $0 \neq a \in C$ is called **prime** in C if

   $a \leq b + c$ and $b, c \in C$ implies that $a \leq b$ or $a \leq c$.

An element $0 \neq a \in C$ is called a **strong atom** if

$b \leq na$ and $0 \neq b \in C, 0 \neq n \in \mathbb{N}_0$ implies that

$b = ma$ with $m \in \mathbb{N}_0, m \leq n$.

(i) Prove that for any cone C and any $0 \neq a \in C$ the following implications hold

a prime $\Rightarrow$ a strong atom $\Rightarrow$ a atom.

(ii) Find counter-examples to the reverse implications in (i). (You may think of the cones in 1(i), 1(ii), 1(iii) or construct a cone by yourself.)

(iii) Draw pictures of the cones in 1(i) and 1(iii) and make a comparison.

(iv) Do you think the notions of an atom and a strong atom possess a geometrical meaning? Explain by drawing pictures.

(v) Can you find a cone with four atoms which are all prime elements? What are the particular features of such a cone?

Lecture 2. Kaplansky geometrical: Krull domains and polyhedra

1. A subcone F of a cone is called a **face** if $x+y \in F$ with $x, y \in C$ implies that $x \in F, y \in F$. A maximal face $\neq C$ of a cone is called a **facet**.

   (i) Find all facets of the cone

   $$C = \{x \in \mathbb{N}_0^3 \mid 2x_1 + 5x_2 = 3x_3\}.$$

   (ii) Show that for a facet F the set $P = C - F = \{x - y \mid x \in C, y \in F\}$ is a halfspace, that is, P is a cone such that

   $$P \cup (-P) = C - C \neq P \qquad (-P = F - C).$$

   (iii) Show that for P as in (ii) it holds that

   $$P \cap (-P) = F - F.$$

   (iv) Draw a picture of the cone as well as of its facets and interprete the relationships in (ii) and (iii) geometrically.

2. Find a cone C in $\mathbb{R}^3$ with $C-C=\mathbb{R}^3$ such that C is the intersection of its halfspaces P (i.e., P is a subcone of C with $P\cup(-P)=\mathbb{R}^3\neq P$) but none of the halfspaces P has the form $P=C-F$ with a facet F of C.

3. Look into I. Kaplansky, Commutative Rings, Revised Edition, The University of Chicago Press 1974

   (i) Try to understand Theorem 1 and its proof on p. 1/2.

   (ii) Try to understand Theorem 2 and its proof on p. 2.