Exercises for

THE ART OF FACTORIZATION

IN MULTIPLICATIVE

STRUCTURES

An MAA PREP Workshop

organized by

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Exercise 1.
Every order in an algebraic number field is an FF*-domain.*

Exercise 2. *The integral domain*

$$R = \mathbb{R} + X\mathbb{C}[X] = \{f \in \mathbb{C}[X] \mid f(0) \in \mathbb{R}\} = \mathbb{R}[X, iX] \subset \mathbb{C}[X]$$

is a one-dimensional noetherian domain with integral closure $\overline{R} = \mathbb{C}[X]$, $(R\colon\overline{R}) \neq \{0\}$ *but* R *is not an* FF*-domain* (*see* [**?**, Example 4.1] *by Anderson-Anderson-Zafrullah, 1990*).

Exercise 3. *Suppose that* H *is atomic but not half-factorial. Then for every* $N \in \mathbb{N}$ *there exists some* $c \in H$ *such that*

$$|\mathsf{L}(c)| \geq N + 1\,.$$

Exercise 4. *The distance function* $\mathsf{d}\colon \mathsf{Z}(H) \times \mathsf{Z}(H) \to \mathbb{N}_0$ *is a metric satisfying* $\mathsf{d}(xz,\, xz') = \mathsf{d}(z,\, z')$ *for all* $x,\, z,\, z' \in \mathsf{Z}(H)$.

Exercise 5. *If* H *is atomic but not factorial, then for every* $k \in \mathbb{N}$ *there exists some* $a \in H$ *such that* $|\mathsf{Z}(a)| \geq k+1$, *and there exist factorizations* $z,\, z' \in \mathsf{Z}(a)$ *such that* $\mathsf{d}(z, z') \geq 2k$.

Exercise 6. *Let* H *be atomic and* $a \in H$.

1. $\mathsf{c}(a) \leq \sup \mathsf{L}(a)$, *and* $\mathsf{c}(a) = 0$ *if and only if* $|\mathsf{Z}(a)| = 1$.
2. *If* $z,\, z' \in \mathsf{Z}(a)$ *and* $z \neq z'$, *then* $2 + \big||z| - |z'|\big| \leq \mathsf{d}(z, z')$.
3. *If* $|\mathsf{Z}(a)| \geq 2$, *then* $2 + \sup \Delta(\mathsf{L}(a)) \leq \mathsf{c}(a)$.
4. *If* $\mathsf{c}(a) \leq 2$, *then* $|\mathsf{L}(a)| = 1$, *and if* $\mathsf{c}(a) \leq 3$, *then* $\mathsf{L}(a)$ *is an arithmetical progression with difference* 1.

Exercise 7. *Suppose that* H *is atomic.*

1. H *is factorial if and only if* $\mathsf{c}(H) = 0$.
2. *If* H *is not factorial, then* $2 + \sup \Delta(H) \leq \mathsf{c}(H)$. *In particular, if* $\mathsf{c}(H) < \infty$, *then* $\Delta(H)$ *is finite.*
3. *If* $\mathsf{c}(H) = 2$, *then* H *is half-factorial.*

4. *If* $\mathsf{c}(H) = 3$, *then every* $L \in \mathcal{L}(H)$ *is an arithmetical progression with difference* 1.

Exercise 8. *Let* R *be a one-dimensional local noetherian domain such that its integral closure* $\overline{R}$ *is a finitely generated* R*-module. Then* $R^\bullet$ *is finitely primary.*

Exercise 9. *Let*

$$H = (\mathbb{N}\times\mathbb{N} \cup \{(0,0)\}, +) \ \subset \ (\mathbb{N}_0^2, +)\,.$$

- H *is finitely primary of rank* 2 *and exponent* 1.
- *If* $\boldsymbol{x} \in H \setminus \{\mathcal{A}(H) \cup \{\mathbf{0}\}\}$, *then* $\min \mathsf{L}(\boldsymbol{x}) = 2$ *and hence* $\rho_2(H) = \infty$.
- $\mathsf{c}(H) = 3$, *whence* $\Delta(H) = \{1\}$ *and all sets of lengths are arithmetical progressions with difference* 1.

Exercise 10. *Apply the above theorem to* $\mathbb{Z}[\sqrt{-7}]$.

Exercise 11.
Let D *be a monoid and* $H \subset D$ *a cofinal submonoid.*

1. *There are epimorphisms*

$$\theta\colon \mathcal{C}(H,D) \to \mathsf{q}(D)/\mathsf{q}(H) \quad \text{and} \quad \theta^*\colon \mathcal{C}^*(H,D) \to \mathsf{q}(D)/D^\times\mathsf{q}(H)\,,$$

given by

$$\theta\big([y]_H^D\big) = [y]_{D/H} = y\mathsf{q}(H) \quad \text{for all} \quad y \in D\,,$$

and

$$\theta^*([y]_H^D\big) = [y]_{D/D^\times H} \quad \text{for all} \quad y \in (D \setminus D^\times) \cup \{1\}\,.$$

2. $[1]_H^D \subset H \subset [1]_{D/H} \cap D$.
3. *The following statements are equivalent*:
   (a) $H \subset D$ *is saturated.*
   (b) $[y]_H^D = [y]_{D/H} \cap D$ *for all* $y \in D$.
   (c) $[1]_H^D = [1]_{D/H} \cap D$.
   (d) $[1]_H^D = H$.

(e) *The epimorphism* $\theta\colon \mathcal{C}(H,D) \to \mathsf{q}(D)/\mathsf{q}(H)$ *defined in* 1. *is an isomorphism.*

Exercise 12. *Let $H \subset D$ be a submonoid.*

1. *If $\mathcal{C}^*(H,D)$ is a group, then $\mathcal{C}(H,D)$ is a group and either $D = D^\times$ or $\mathcal{C}(H,D) = \mathcal{C}^*(H,D)$.*
2. *If $\mathcal{C}(H,D)$ is a group, then $H \subset D$ is cofinal, and if $\mathcal{C}(H,D)$ is a torsion group, then $H \subset D$ is also saturated.*

Exercise 13. *Let $k, l \in \mathbb{N}$.*

1. *We have*
$$\rho(G) = \sup\Big\{\frac{\rho_m(G)}{m}\ \Big|\ m \in \mathbb{N}\Big\} = \lim_{m\to\infty} \frac{\rho_m(G)}{m},$$
$$k \le \rho_k(G) \le k\rho(G) = \frac{k\mathsf{D}(G)}{2}, \quad \rho_{2k}(G) = k\mathsf{D}(G)$$
2. $k + l \le \rho_k(G) + \rho_l(G) \le \rho_{k+l}(G)$.
3. *For all $k \in \mathbb{N}$ we have*
$$1 \le \rho_{2k+1}(G) - k\mathsf{D}(G) \le \Big\lfloor \frac{\mathsf{D}(G)}{2} \Big\rfloor.$$
4. *Let $m \in \mathbb{N}$ with*
$$\rho_{2m+1}(G) - m\mathsf{D}(G) = \max\{\rho_{2k+1}(G) - k\mathsf{D}(G) \mid k \in \mathbb{N}\}.$$
Then
$$\rho_{2m+2i+1}(G) = \rho_{2m+1}(G) + i\mathsf{D}(G) \quad \text{for all} \quad i \in \mathbb{N}_0.$$
In particular,
$$\rho_3(G) \ge \Big\lfloor \frac{3\,\mathsf{D}(G)}{2} \Big\rfloor \quad \text{implies} \quad \rho_k(G) = \Big\lfloor k\frac{\mathsf{D}(G)}{2} \Big\rfloor \quad \text{for all} \quad k \ge 2.$$

Exercise 14. *Let $|G| \ge 3$.*

1. $1 \in \Delta^*(G) \subset \Delta(G) \subset [1, \mathsf{D}(G) - 2]$.
2. *If there exists some $g \in G$ with $3 \le \operatorname{ord}(g) < \infty$, then $\operatorname{ord}(g) - 2 \in \Delta^*(G)$.*
3. *If $\mathsf{r}(G) \ge 2$, then $[1, \mathsf{r}(G) - 1] \subset \Delta^*(G)$.*
4. *There exists a minimal non-half-factorial subset $G_0 \subset G$ with $\max \Delta^*(G) = \min \Delta(G_0)$.*

Exercise 15.

1. $\mathcal{L}(G) = \{y + L \mid y \in \mathbb{N}_0,\ L \in \mathcal{L}(G \setminus \{0\})\} \supset \big\{\{y\} \bigm| y \in \mathbb{N}_0\big\}$, *and equality holds if and only if* $|G| \le 2$.
2. *If* $G_0 \subset G$ *is a subset, then* $\mathcal{L}(G_0) \subset \mathcal{L}(G)$.
3. *Let* G' *be an abelian group with* $|G'| \ge 3$ *such that* $\mathcal{L}(G) = \mathcal{L}(G')$. *Then we have* $\rho_k(G) = \rho_k(G')$ *for every* $k \in \mathbb{N}$, $\mathsf{D}(G) = \mathsf{D}(G')$, $\Delta_1(G) = \Delta_1(G')$ *and* $\max \Delta^*(G) = \max \Delta^*(G')$.
4. *There exist* (*up to isomorphisms*) *only finitely many abelian groups* G' *such that* $\mathcal{L}(G) = \mathcal{L}(G')$, *and all of them are finite.*