Chapter 3

Extensions of Half-Factorial Domains: A Survey

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Abstract

A half-factorial domain (HFD) is an atomic domain, R, with the property that if one has the irreducible factorizations in R

$$\pi_1\pi_2\cdots\pi_n = \xi_1\xi_2\cdots\xi_m$$

then $n = m$.

Half-factorial domains were implicitly studied in the case of rings of algebraic integers in a 1960 paper of L. Carlitz [11] and were subsequently abstracted and studied by A. Zaks.

Since their inception as a generalization of the classical notion of unique factorization domains, half-factorial domains have been the subject of much interest in commutative algebra. This paper will give a survey of some recent advances in the study of half-factorial domains with the emphasis on advances in the understanding of ring extension behavior of half-factorial domains.

1 Introduction and a bit of history

The purpose of this paper is to give an expository overview of some of the recent work in the study of half-factorial domains and their extension rings. This paper is written as an expansion on an invited 45-minute address presented in Chapel Hill in October of 2003. The author would like to begin by expressing his gratitude to Professors Scott Chapman and Bill Smith for their kind invitation and hospitality.

The concept of "half-factorial domain" was introduced implicitly in 1960 in a striking paper of L. Carlitz [11]. The paper was written as an answer to a challenge of Narkiewicz who asked for a purely arithmetic characterization of rings of integers in terms of the class number. Of course, it was well-known that a ring of algebraic integers, and more generally, a Dedekind domain, has *unique* factorization if and only if its class number is one. Given this interplay between the concepts of unique factorization and class number (and class group), Narkiewicz' line of questioning was most natural.

In 1960, Carlitz' partial answer to this question appeared in the Proceedings of the American Mathematical Society. In our opinion, this result is one of the most succinct and beautiful results in the study of factorization in rings of integers. It is also, from at least one point of view, the paper that is the ancestor of much of the current study in the general theory of factorization in general integral domains. Additionally, this paper is one of the most cited papers in factorization theory, and the arguments utilized in the proof have made a "factorization industry" possible for large classes of domains. The proofs in the paper underscore the "right" direction in which to proceed in the more general case.

The main result of this seminal paper is that if R is a ring of algebraic integers then one has unique *length* of factorizations if and only if the class number does not exceed 2. Because of the importance of this result and the fundamental nature of its proof, we will state this result (and demonstrate its proof) more formally later. In the interim, we will remind the reader of the "nicest" class of domains from a factorization point of view and give the current definition of a half-factorial domain as a generalization. This not only gives the historical line of reasoning, but also makes the statement of Carlitz' result much cleaner.

The result of Carlitz' 1960 paper remained mostly untouched for about 15-20 years until Zaks generalized this "unique length of factorization" property. He published a couple of papers [47, 48] on the subject and coined the terminology "half-factorial domain." The paper [48] is a longer work with a number of important theorems on the subject of the half-factorial property. It is interesting to note that what seems to be the current accepted definition of a half-factorial domain is actually different from the one in Zaks' paper.

About 10 years later the real gold-rush began with a paper by D. D. Anderson, D. F. Anderson, and M. Zafrullah [2]. This paper (and its sequel [3]) is a blueprint for much of the work that is taking place in modern factorization theory, and the scope of the paper far exceeded the scope of the half-factorial property. Historically, this marks the time when much activity began in the factorization theatre.

In this paper, we will concentrate on work that has been done in the theory of half-factorial domains and their extension rings. Because of the large number of results obtained in various aspects of factorization, this paper will be far from complete. The interested reader would be well-served to consult the volume [1] for a very good collection of papers on the half-factorial property, its generalizations, and factorization in general.

In this paper we use the following notation conventions:

1) R is (unless otherwise stated) an atomic domain with quotient field K.

2) R_S is the localization of R at a multiplicative set S.

3) $\overline{R}$ is the integral closure of R.

4) R' is the complete integral closure of R.

5) $U(R)$ is the unit group of R.

6) $Irr(R)$ is the set of irreducible elements of R.

7) $Cl(R)$ is the class group of R.

8) $R[x]$ and $R[[x]]$ are the rings of polynomial and formal power series over R, respectively.

9) $D(G)$ is the Davenport constant of the finite abelian group G.

10) $d(K)$ is the discriminant of the algebraic number field K.

11) S_n is the symmetric group on n letters.

Other notations are standard.

2 Some preliminary results and definitions

Of course, the genesis of factorization theory is motivated by the domains where factorization is extremely well-behaved. For the purposes of this paper, we will assert some control over the "badness" of the factorization behavior. That is, unless otherwise stated, we will assume that our domains are *atomic* (i.e., every nonzero, nonunit element of R can be written as a product of irreducible elements).

We now recall the definition of unique factorization domains.

Definition 2.1. *A unique factorization domain (UFD) is an atomic integral domain, R, such that given the irreducible factorizations*

$$\pi_1\pi_2\cdots\pi_n = \xi_1\xi_2\cdots\xi_m$$

we have that:

a) $n = m$ *and*

b) $\exists\ \sigma \in S_n$ *such that for each* $1 \leq i \leq n$, $\pi_i = u_i\xi_{\sigma(i)}$ *for some* $u_i \in U(R)$.

There are a number of equivalent definitions of UFDs that are useful in different settings. In particular, a UFD may be more succinctly defined as the class of domains where every nonzero nonunit can be expressed as a product of prime elements. From the ideal theoretic point of view, UFDs can also be characterized as the class of domains where every nonzero prime ideal contains a nonzero principal prime [32]. Unique factorization domains possess a large number of very nice properties and we will compare and contrast some of these with the properties of half-factorial domains.

The class of half-factorial domains (HFDs) is, from one point of view, the nicest generalization of the class of UFDs with respect to factorization.

Definition 2.2. *An (atomic) domain is said to be a half-factorial domain (HFD) if given two irreducible factorizations*

$$\pi_1\pi_2\cdots\pi_n = \xi_1\xi_2\cdots\xi_m$$

then $n = m$.

So, in a certain sense, an HFD is just a UFD with "half the axioms."

It is worth noting that Zaks' original definition omitted the "atomic" assumption (and in Zaks' paper he quickly specialized to atomic cases). Of course, it is natural to ask if the HFD condition without the atomic assumption automatically forces atomicity, but this is not the case as we will see in the next example. Without the "atomic" assumption, there is some pathology that we wish to avoid; for this reason (and since today's common definition also includes the assumption "atomic"), we will always assume that HFD implies atomic unless otherwise indicated.

In the following example from [25], we will produce a nonatomic domain that has the property that any two equal irreducible factorizations have the same length.

Example 2.3. *This example will give a nonatomic domain with the property that if*

$$\pi_1\pi_2\cdots\pi_n = \xi_1\xi_2\cdots\xi_m$$

with $\pi_i, \xi_j \in Irr(R),\ 1 \leq i \leq n,\ 1 \leq j \leq m$*, then* $n = m$*.*

The example is a domain with a unique nonprime irreducible element (up to associates) and was constructed in [25]. In this domain, the uniqueness of the irreducible assures that any two irreducible factorizations are of the same length (in fact, unique). The ease of this observation belies the delicate nature of the construction of a domain with a unique nonprime irreducible (for atomic domains with finitely many irreducibles, the so-called CK domains, see [4]). Speaking very loosely, we will outline the major steps to the construction of such a domain.

Begin with a ring R *that possesses a nonprime* $\pi \in Irr(R)$*. One then constructs a new domain,* R_1*, by indexing all the irreducibles in* R *that are not associate to* π *(say our relevant irreducibles are* $\{\xi_i\}_{i\in I}$ *with* I *some indexing set) and building a larger domain that "forces" each* ξ_i *to become reducible. In particular, we let*

$$R_1 = R[\{x_i\}][\frac{\xi_i}{x_i}]_{i\in I}.$$

with each x_i *a polynomial indeterminate.*

It is then shown that in R_1*,* π *remains a nonprime irreducible and each* ξ_i *is reducible in* R_1 *by construction. Of course, there may be many "new irreducibles", but one again isolates* π *and constructs* R_2 *analogously. Continuing in this fashion, we obtain a tower of rings*

$$R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots$$

and we define $T := \bigcup_{i=1}^{\infty} R_i$*.*

It can be shown that in T*,* π *is the unique nonprime irreducible (intuitively, uniqueness comes from the fact that every possible irreducible not associate to* π *is "destroyed" in the next stage of the construction).*

This domain has the claimed property, and it is interesting to note that this construction shows that any domain with a nonprime irreducible element can be embedded in such a domain.

We remark that the general theme of this construction is very much in the spirit of the construction performed by M. Roitman in [42] where it was shown that a polynomial extension of an atomic domain is not necessarily atomic.

With these definitions in hand, we now have the tools to succinctly state the classical result of Carlitz. We give a slightly more general statement in the spirit of the general utility of Carlitz' proof and in the spirit of much of the research that has taken place concerning the interplay of the class number and factorization.

Theorem 2.4. *Let D be a Dedekind domain. If $|Cl(D)| \leq 2$ then D is an HFD. If $Cl(D)$ is torsion and D has the property that there is a prime ideal in every class, then the converse also holds.*

Proof. Of course, it is well-known that if $Cl(D)$ is trivial, then D is a UFD (and hence an HFD), so we will concentrate on the case where $|Cl(D)| = 2$. Assume that we have the irreducible factorizations

$$\pi_1\pi_2\cdots\pi_n = \xi_1\xi_2\cdots\xi_m.$$

Since a prime factor appears on the left side if and only if it appears on the right, we will reduce to the case where the factorizations above consist entirely of nonprime irreducible elements. Since $|Cl(D)| = 2$ and each irreducible is a nonprime we can factor (in terms of prime ideals)

$$\pi_i = \mathfrak{P}_1^{(i)}\mathfrak{P}_2^{(i)}$$

and

$$\xi_i = \mathfrak{Q}_1^{(i)}\mathfrak{Q}_2^{(i)}.$$

We now have the ideal factorization

$$\mathfrak{P}_1^{(1)}\mathfrak{P}_2^{(1)}\mathfrak{P}_1^{(2)}\mathfrak{P}_2^{(2)}\cdots\mathfrak{P}_1^{(n)}\mathfrak{P}_2^{(n)} = \mathfrak{Q}_1^{(1)}\mathfrak{Q}_2^{(1)}\mathfrak{Q}_1^{(2)}\mathfrak{Q}_2^{(2)}\cdots\mathfrak{Q}_1^{(m)}\mathfrak{Q}_2^{(m)}.$$

Since prime ideal factorizations are unique, we have that $2n = 2m$ and hence $n = m$.

On the other hand, assume that D is a Dedekind HFD and $Cl(D)$ has a prime in every class. Since $Cl(D)$ is torsion, every class has finite order. We first assume the existence of a class of order $n > 2$ and select a prime ideal $\mathfrak{P}$ in this class. By construction, $\mathfrak{P}^n$ is principal and generated by some $\alpha \in Irr(D)$. We now choose a prime ideal $\mathfrak{Q}$ in the class of $\mathfrak{P}^{-1}$ and note that $\mathfrak{P}\mathfrak{Q}$ is principal and generated by some $\pi \in Irr(D)$ and additionally $\mathfrak{Q}^n$ is principal and generated by the irreducible β. We now consider the ideal factorizations

$$(\mathfrak{P}\mathfrak{Q})^n = (\mathfrak{P}^n)(\mathfrak{Q}^n)$$

which give the elemental factorizations

$$\pi^n = u\alpha\beta$$

for some $u \in U(D)$. Since $n > 2$, D is not an HFD.

The only case left to consider is the case where $Cl(D)$ is of exponent 2 and $|Cl(D)| > 2$. In this case, $Cl(D)$ must contain a subgroup isomorphic to the Klein 4-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In a fashion similar to the above, we choose prime ideals $\mathfrak{P}$ in the class corresponding to $(0, 1)$, $\mathfrak{Q}$ in the class corresponding to $(1, 0)$ and $\mathfrak{R}$ in the class corresponding to (1,1). It is easy to see that there are irreducibles α, β, γ, and ξ such that $\mathfrak{P}^2 = (\alpha)$, $\mathfrak{Q}^2 = (\beta)$, $\mathfrak{R}^2 = (\gamma)$, and $\mathfrak{P}\mathfrak{Q}\mathfrak{R} = (\xi)$. The ideal factorizations

$$(\mathfrak{P}\mathfrak{Q}\mathfrak{R})^2 = (\mathfrak{P}^2)(\mathfrak{Q}^2)(\mathfrak{R}^2)$$

give rise to the elemental factorizations

$$\xi^2 = u\alpha\beta\gamma$$

for some $u \in U(R)$ and once again D is not an HFD. ∎

A couple of remarks are worth noting here. First, the original Carlitz result was couched strictly in terms of algebraic number theory. His result was that a ring of algebraic integers, R, is an HFD if and only if the class number of R does not exceed 2. But the seeds of the original proof pointed the way towards much more generality.

The second remark is that although the above theorem is slightly more general than the original statement of Carlitz, it is not as general as possible either. There has been much work done in more general cases and the interested reader is encouraged to consult [5, 6, 14].

3 "Good" properties of UFDs versus HFDs

As a generalization of UFDs, one would like to understand what, if any, "nice" behaviors of UFDs are shared by HFDs. UFDs possess a number of nice properties (especially in the case of localizations and polynomial extensions). Some of these classical results are that if R is a UFD then so is $R[x]$, and the localization R_S. Both of these central results for UFDs can be obtained from the following nice ideal-theoretic characterization of UFDs.

Theorem 3.1. *A domain R is a UFD if and only if every nonzero prime ideal in R contains a (nonzero) prime element.*

So a UFD is the "multi-dimensional" generalization of a principal ideal domain (PID). Generally speaking, HFDs do not have a readily apparent ideal theoretic characterization. In the case of rings of algebraic integers (and more generally, Dedekind and Krull domains with certain restrictions on the class group), HFDs *do* have a nice ideal-theoretic characterization (evidenced by Carlitz' theorem, for one). A sweeping ideal-theoretic characterization for general HFDs seems to be elusive (if indeed such a characterization exists), and we cannot resist the following question.

Question 3.2. *Find a general ideal-theoretic characterization of HFDs in the spirit of the characterization of UFDs, if it exists.*

Below we produce a table that records some of the classical results about UFDs and compares them with what is known about the analog property for HFDs. It should be explicitly stated that the entries "no" in the table mean, more precisely, "not always."

Table of Inherited Factorization Properties

R is a:	$R[x]$	$R[[x]]$	$\overline{R}$	R'	R_S
UFD	Yes	No	Yes	Yes	Yes
HFD	No	No	No	No	No

We make a couple of remarks about the table. First it should be noted that if R is a UFD then the (complete) integral closure of R coincides with R, so this result is trivial (but it does indeed invite the question as to whether the (complete) integral closure of an HFD is still an HFD). Also, although it is possible for a power series ring over a UFD to lose unique factorization [43], if R is a one dimensional UFD (i.e., a PID) then $R[[x_1, x_2, \cdots, x_n]]$ is a UFD for all n.

If the reader hopes for the half-factorial property to be preserved in standard extensions, then this table will look quite depressing. However, we remark that there are indeed positive results with (in some cases) rather minimal restrictions places on the HFD. We will look at some of the positive results and some of the pathologies in the next sections.

4 Localizations

In our view, one of the most disturbing results in the table from the previous section is that if R is an HFD, then it is not necessarily true that R_S is an HFD (where S is a multiplicatively closed subset of R). The proof that the UFD property holds in localizations is very elementary (the characterization of UFD via every nonzero prime contains a nonzero principal prime is very useful in this regard). Of course, one would not expect general overrings to preserve the half-factorial property since unique factorization is not preserved in general overrings. A quick example of this is the ring $K[x, y]$ where K is a field. This ring is a UFD (and hence an HFD), but the overring $K[x, \frac{y}{x}, \frac{y}{x^2}, \frac{y}{x^3}, \cdots]$ is not even atomic.

In this section we take a brief look at some examples, positive (and negative) results, and generalizations of HFDs in the case of localizations and more general overrings. We will begin by giving an example of an HFD which possesses a localization which does not have the half-factorial property. This example and other very nice results in this spirit may be found in the papers [7], [8], and [9].

Example 4.1. *We begin by noting that one can construct a Dedekind domain, D, with class group $\mathbb{Z}_6$ such that the prime ideals of D are restricted to the classes corresponding to $\{1, 2, 3\} \subseteq \mathbb{Z}_6$ (a more general theorem on the construction of such Dedekind domains can be found in [27]). As it turns out, such a Dedekind domain is necessarily an HFD.*

We now consider the set

$$\Gamma = \{Q | Q \text{ is a prime ideal of } D \text{ with } [Q] = 1 \text{ or } [Q] = 2\}.$$

It is easy to see that if P is a prime ideal such that $[P] = 3$ then $P \not\subseteq \bigcup_{Q\in\Gamma} Q$. Let $t \in P \setminus \cup_{Q\in\Gamma} Q$, $T = \{1, t, t^2, \cdots\}$ and form the localization $R = D_T$. We make the following observations.

1. *R is a Dedekind domain.*

2. $Cl(R) \cong Cl(D)/(ker(\tau))$ where τ is the natural map from $Cl(D) \longrightarrow Cl(R)$ defined by $\tau : [I] \longrightarrow [IR]$.

It follows ([7]) that $Cl(R) \cong \mathbb{Z}_6/\mathbb{Z}_2 \cong \mathbb{Z}_3$ and the primes of R are in the nontrivial classes of $\mathbb{Z}_3$. With this reduction, the technique of the Carlitz proof can be applied to show that R is not an HFD.

This example motivates the following definitions (once again from [7] and [8].

Definition 4.2. *Let R be an integral domain. We say that R is a*

a) Locally half-factorial domain (LHFD) if every localization of R is an HFD.

b) Strong half-factorial domain (SHFD) if every overring of R is an HFD.

Of course, an SHFD is an LHFD, but the converse does not hold in general. Here are some nontrivial examples of these types of domains.

Example 4.3. *Let R be a Dedekind domain with $Cl(R) = \mathbb{Z}_p$ for a prime integer p, such that all primes of R are contained in the class $\{1\}$. Then it can be shown that any (nontrivial) localization R_S has trivial class group and hence is a PID. The upshot is that R is a LHFD, and since R has torsion class group, every overring is a localization and hence R is SHFD.*

We note that this example can be tweaked to the case where $Cl(R)$ is not torsion to give an example of an LHFD that is not an SHFD (the key in this augmentation of the example is that if the class group is not torsion, then there are overrings which are not localizations).

Theorem 4.4. *Let D be a Dedekind domain satisfying one of the following.*

a) $|Cl(D)| \leq 5$.

b) $Cl(D) = \mathbb{Z}_{p^n}$.

c) $Cl(D) = \oplus_{i=1}^{k} \mathbb{Z}_2$.

Then the following statements are equivalent.

1. D is an HFD.

2. D is a SHFD.

3. D is a LHFD.

We conclude this brief section with a couple of observations. The first is that the SHFD property is very strong in the sense that if R is an SHFD then the Krull dimension of R is no more than 1 (the introductory example at the beginning of the section of the overring pair $K[x, y] \subseteq K[x, \frac{y}{x}, \frac{y}{x^2}, \frac{y}{x^3}, \cdots]$ illustrates very plainly the hazards of domains where the Krull dimension is 2 or more).

Finally, if G is a finite abelian group NOT of the form a), b), or c) in the theorem, then there exists a Dedekind domain D with $Cl(D) = G$ such that D is not an HFD but every proper overring is an HFD.

Some interesting generalizations of HFDs (e.g., "congruence" HFDs and k-HFDs) are studied in [15].

5 Polynomial extensions

One of the most famous and useful properties of UFDs is the fact that unique factorization in a coefficient ring R makes a smooth transition to the polynomial ring $R[x]$. This result, originally due to Gauss, makes polynomial computations (over a well-behaved coefficient ring) quite tractable.

Theorem 5.1. *If R is a UFD, then $R[x]$ is a UFD.*

The standard proof of this theorem uses the notion of "content" of a polynomial (recall that the content of a polynomial $f \in R[x]$ is the ideal of R generated by its coefficients), but a more streamlined proof can be given using the ideal-theoretic characterization of UFDs (any nonzero prime contains a nonzero prime element).

This theorem is a classic and begs the analogous question for HFDs. In [48], Zaks proved the following generalization of this theorem to a special class of HFDs.

Theorem 5.2. *If R is a Krull domain, then $R[x]$ is an HFD $\Longleftrightarrow$ $|Cl(R)| \leq 2$.*

Unfortunately, Zaks' result does not generalize to all HFDs as the following example will show.

Example 5.3. *Consider the order $R := \mathbb{Z}[\sqrt{-3}] \subseteq \mathbb{Z}[\sqrt{\omega}]$ where $\omega = \frac{1+\sqrt{-3}}{2}$. The fact that R is an HFD is easily shown (for example, see [48]). Intuitively, $\mathbb{Z}[\sqrt{-3}]$ is an HFD because the integral closure, $\mathbb{Z}[\omega]$, is a UFD and a norm argument shows that every irreducible in $\mathbb{Z}[\sqrt{-3}]$ is prime in $\mathbb{Z}[\omega]$.*

Now consider the factorization

$$(2x - 2\omega)(2x - 2\overline{\omega}) = (2)(2)(x^2 - x + 1)$$

where $\overline{\omega}$ is the complex conjugate of ω.

Clearly, the right hand side of the above equation has at least three irreducible factors (in fact, precisely three), and so if both of the factors on the left are irreducible, then $\mathbb{Z}[\sqrt{-3}][x]$ is not an HFD. And indeed this is the case, as any irreducible factorization of $2x - 2\omega$ (respectively $2x - 2\overline{\omega}$) must consist of a constant in $\mathbb{Z}[\sqrt{-3}]$ of norm 2 or 4, and a degree one polynomial factor. It is easy to check that there is no such factorization.

Although the above example demonstrates the impossibility of a full generalization of Gauss' theorem to general HFDs, it does provide some insight. A slight tweaking of this example to a slightly more general setting allows us more to glean the following theorem [17].

Theorem 5.4. *If $R[x]$ is an HFD, then R is an integrally closed HFD.*

The proof of this theorem may be found in [17], but the example above holds almost the entire content of this theorem. The interested reader is encouraged to "play" with the above example; it is easily seen that the failure of the half-factorial property in $\mathbb{Z}[\sqrt{-3}]$ is a direct consequence of the fact that the coefficient ring is not integrally closed.

As a corollary to the previous theorem (coupled with the results from [48]), one can classify all Noetherian polynomial HFDs.

Corollary 5.5. *Let R be Noetherian. The following conditions are equivalent.*

a) $R[x]$ *is an HFD.*

b) $R[x_1, x_2, \cdots, x_n]$ *is an HFD for all* $n \geq 1$.

c) R *is a Krull domain with* $|Cl(R)| \leq 2$.

Proof. It is known from [48] that if R is a Krull domain, then $R[x]$ is an HFD if and only if $|Cl(R)| \leq 2$, which gives c) implies a).

For a) implies c), we assume that $R[x]$ is an HFD. This means that R is integrally closed, and since R is Noetherian, R is a Krull domain. We apply Zaks' result again to obtain c), and hence a) and c) are equivalent.

Certainly b) implies a) and to get the converse, we merely note that if R is a Krull domain, then so is $R[x_1, \cdots, x_n]$ and what is more, the class number is stable upon the adjunction of any finite number of indeterminates. ∎

6 Power series extensions

Power series extensions are oftentimes more problematic than polynomial extensions. Many classical results that hold (in general commutative algebra) sometimes fail wildly in the setting of power series. Passing to a completion (even an $x-$adic one) is sometimes a bit tricky and some nice properties may be lost.

For the sake of perspective, a striking example of this phenomenon is in dimension theory. A classical result for polynomials is that if the (Krull) dimension of a ring ($\dim(R)$) is finite (say $\dim(R) = n$), then so is the dimension of $R[x]$ (in particular, if $\dim(R) = n$ then $n + 1 \leq \dim(R[x]) \leq 2n + 1$). This is wildly untrue (and in fact, from the non-Noetherian point of view, usually untrue) in the case of power series rings. In fact, there are $0-$dimensional rings whose power series extensions are infinite dimensional. Of course, it should be noted that there are instances in which the behavior of formal power series is at least as nice as the analog behavior in polynomials. One example where this occurs is in the case of the passage of the unit group of a ring to polynomials and power series. It is well-known that $U(R) = U(R[x])$ and that the set of units in $R[[x]]$ is the set of power series $f(x) \in R[[x]]$ such that $f(0) \in U(R)$.

An even more striking example of good power series behavior involves (semi-)quasi-local rings. It is a central result that R is quasi-local (resp., semi-quasi-local) if and only if $R[[x]]$ is quasi-local (resp. semi-quasi-local). The analogous result for polynomials is not true, since the ring $R[x]$ is *never* semi-quasi-local. But such nice behavior of power series rings relative to the polynomial case is the exception and not the rule in practice.

From a factorization point of view we can find bad behavior as well. For example, there are UFDs which have non-UFD power series extensions [43].

As has been pointed out earlier, if $R[x]$ is an HFD, then R is integrally closed. Since for the polynomial case, the coefficient ring being integrally closed is necessary for $R[x]$ to have a chance at the half-factorial property, intuitively one would (perhaps) expect that $R[[x]]$ being an HFD would demand at least this much. In light of this "intuition" let us revisit an earlier example.

Example 6.1. *Consider the order* $R := \mathbb{Z}[\sqrt{-3}] \subseteq \mathbb{Z}[\sqrt{\omega}]$ *where* $\omega = \frac{1+\sqrt{-3}}{2}$*. As noted before,* R *is an HFD with* $R[x]$ *failing to be an HFD. We again look at the factorization in* $\mathbb{Z}[x]$ *that demonstrated the loss of the half-factorial property:*

$$(2x - 2\omega)(2x - 2\overline{\omega}) = (2)(2)(x^2 - x + 1).$$

A close inspection of this factorization shows that, in contrast to the $\mathbb{Z}[x]$ *case, this does not deny the half-factorial property in* $\mathbb{Z}[[x]]$*. The reason for this is that the element* $x^2 + x + 1$*, although an irreducible in* $\mathbb{Z}[x]$*, is a unit in* $\mathbb{Z}[[x]]$*. Hence up to units, the factorizations on both sides of the above equation are of length 2.*

In fact, in the example above, $R[[x]]$ is, in fact, an HFD. Much more general situations are considered in [22]. We will outline a proof that the above example is a (non-integrally closed) example of an HFD such that $R[[x]]$ is an HFD. A more thorough treatment of this phenomenon can be found in [22].

Theorem 6.2. *Let* R *be a* $1-$*dimensional domain with integral closure* $\overline{R}$ *and conductor* I*. Also suppose that every nonzero coset of* $\overline{R}/I$ *can be written in the form* $u + I$ *with* $u \in U(\overline{R})$*. Then* $\overline{R}$ *is a UFD implies that* $R[[x]]$ *is an HFD.*

Before beginning this proof, we note that, in particular, the hypotheses apply to the ring $\mathbb{Z}[\sqrt{-3}]$. Indeed, the integral closure of this ring is (the UFD) $\mathbb{Z}[\omega]$ where $\omega = \frac{1+\sqrt{-3}}{2}$ and the conductor of $\mathbb{Z}[\omega]$ to $\mathbb{Z}[\sqrt{-3}]$ is the ideal $2\mathbb{Z}[\omega] = \mathfrak{P}$. Since this conductor ideal is prime, the quotient ring $\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]$ is isomorphic to $\mathbb{F}_4$, the field of 4 elements. It is easy to see that the nonzero cosets can be written in the form $1 + \mathfrak{P}, \omega + \mathfrak{P}$, and $\omega^2 + \mathfrak{P}$.

We will now give an outline of the proof of the theorem.

Proof. We claim that every irreducible element of $R[[x]]$ is again irreducible in $\overline{R}[[x]]$. If not, then we factor an irreducible $f \in R[[x]]$ as gh with $g, h \in \overline{R}[[x]]$. First note that if both h and g are in $I[[x]]$, then this is a direct contradiction. We begin by assuming that *neither* g nor h is an element of $I[[x]]$; we write

$$g(x) = a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + x^k(b_k + b_{k+1}x + \cdots)$$

where b_k is the first term not in the conductor I. We write $b_k = u_1 + I_g$ with u_1 a unit of $\overline{R}$ and $I_g \in I$. Collapsing notation we write

$$g = \overline{g} + x^k(u_g + I_g)$$

where u_g is a unit power series with constant coefficient $u_1 \in U(\overline{R})$ and $\overline{g} \in I[[x]]$.

In a similar fashion, we write

$$h = \overline{h} + x^m(u_h + I_h)$$

with u_h a unit power series with constant coefficient $v_1 \in U(\overline{R})$, $\overline{h} \in I[[x]]$, and $I_h \in I$.

Note that

$$\begin{aligned} gh =& \overline{g}\overline{h} + \overline{g}x^m(u_h + I_h) + \overline{h}x^k(u_g + I_g) + \\ &+ x^{k+m}(u_gu_h + u_gI_h + u_hI_g + I_gI_h). \end{aligned}$$

Since $\overline{h}, \overline{g}, I_g$, and I_h are in $I[[x]]$ and g and h are in $R[[x]]$, we obtain that $u_g u_h$ is an element of $R[[x]]$. As a consequence we note that since

$$g = \overline{g} + x^k(u_g + I_g)$$

we have that

$$u_h g = u_h\overline{g} + x^k(u_h u_g + u_h I_g)$$

and so $u_h g \in R[[x]]$. Similarly, we obtain that $u_g h \in R[[x]]$.

We now note that

$$f = gh = (u_h g)(u_g h)(u_h u_g)^{-1}$$

and we have a contradiction. The final case to consider is the case when precisely one of the factors, g or h, is an element of $I[[x]]$. We will assume without loss of generality that $g = \overline{g}$ is the factor in $I[[x]]$. In this case, $u_h g \in R[[x]]$ and we have

$$f = gh = (u_h g)(u_h^{-1} h)$$

which is again a contradiction.

Now that we have established that every irreducible in $R[[x]]$ is irreducible in $\overline{R}[[x]]$, we observe that if we have two irreducible factorizations in $R[[x]]$

$$f_1 f_2 \cdots f_n = g_1 g_2 \cdots g_m$$

then each f_i, g_j is irreducible in $\overline{R}[[x]]$ which is a UFD (since $\overline{R}$ is a 1 dimensional UFD and hence a PID). Hence $n = m$. ∎

We cannot resist highlighting what we believe to be an interesting implication of this example.

Corollary 6.3. *There exist HFDs R such that the half-factorial property is lost in $R[x]$ and regained in $R[[x]]$.*

Proof. If one considers the ring $R := \mathbb{Z}[\sqrt{-3}]$, then this is an HFD such that $R[x]$ is not an HFD, since R is not integrally closed. Nonetheless, the above result shows that $R[[x]]$ is indeed an HFD. ∎

Here is a final observation along these lines.

Corollary 6.4. *If $R[[x]]$ is a UFD, then R and $R[x]$ are UFDs. If $R[[x]]$ is an HFD, then R is an HFD, but $R[x]$ is not necessarily an HFD.*

Proof. First we note that any irreducible element of R remains irreducible in $R[[x]]$. Indeed, if $\pi \in Irr(R)$ and $\pi = f(x)g(x)$ with $f(x), g(x) \in R[[x]]$ then we write $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_i x^i$ and factor

$$\pi = (\sum_{i=0}^{\infty} a_i x^i)(\sum_{i=0}^{\infty} b_i x^i).$$

It is immediate that we get the factorization

$$\pi = a_0 b_0$$

and since $\pi \in Irr(R)$, either a_0 or b_0 is a unit in R, forcing either $f(x)$ or $g(x)$ to be a unit in $R[[x]]$. This establishes our claim.

Given this claim, we consider the following factorization in R

$$\pi_1 \pi_2 \cdots \pi_n = \xi_1 \xi_2 \cdots \xi_m$$

with each $\pi_i, \xi_j \in Irr(R)$.

If $R[[x]]$ is an HFD, then all of the above irreducible elements remain irreducible in $R[[x]]$ and since $R[[x]]$ is an HFD, we must have $n = m$ and R is an HFD.

Additionally, if $R[[x]]$ is a UFD, then every irreducible element of $R[[x]]$ is prime, and hence each π_i and ξ_j is a prime element of $R[[x]]$ (and it is easy to see that the elements are therefore prime as elements of R). Hence R is a UFD.

We now have that if $R[[x]]$ is a UFD (respectively HFD) then R is a UFD (respectively HFD). The fact that if $R[[x]]$ is a UFD implies that $R[x]$ is a UFD follows from the previously mentioned result of Gauss, and the absence of the analogous result for HFDs is demonstrated by the above example. ∎

We remark here that one might notice that there is a similar result for the case of (semi)quasi-local rings. That is, if $R[[x]]$ is (semi)quasi-local then R is (semi)quasi-local, but $R[x]$ is not. One thing to contrast, however, is that $R[x]$ is *never* (semi)quasi-local.

As was noted before, the above technique is used with some success in [22] in a more general setting. In the previous section, a classification of all Noetherian polynomial HFDs was given, but a general classification for the power series case (even for Noetherian rings) remains elusive. In hindsight, this might not be too surprising since a good characterization of when the UFD property is preserved in power series extensions is not known. One of the best theorems in this sense is the result that states that if R is a PID then $R[[x_1, x_2, \cdots, x_n]]$ is a UFD for all $n \geq 1$. The class of PIDs is precisely the class of one-dimensional UFDs and it should be noted that the classical example of a UFD, R, such that $R[[x]]$ is not a UFD is a two-dimensional Noetherian domain. This demonstrates, in a certain sense, that one does not have to travel far from the class of PIDs to find examples of bad factorization behavior in power series extensions.

7 Extensions of the form $A + xB[x]$ and $A + xI[x]$

A key source of examples in commutative algebra is the so-called "$D + \mathfrak{M}$" construction. Recently the $D + \mathfrak{M}$ construction has shown that its utility extends to the study of factorization (often as a rich source of "bad" factorization behavior). A useful illustration of this is the ring $\mathbb{Z} + x\mathbb{Q}[x]$. This ring is one of the simplest examples of the previous statement. It is non-atomic, since the element x is a nonunit that possesses no finite factorization into irreducibles. To see this, we note that if

$$x = \pi_1 \pi_2 \cdots \pi_n$$

is an irreducible factorization of x in $\mathbb{Z} + x\mathbb{Q}[x]$, then a degree argument, coupled with the fact that $\mathbb{Q}[x]$ is a PID shows that precisely one of the irreducible factors (say π_1 without loss of generality) must be an associate of x as an element of $\mathbb{Q}[x]$. We write $\pi_1 = qx$ with $q \in \mathbb{Q}$ and note that $\pi_1 = 2(\frac{q}{2})x$, which contradicts the irreducibility of π_1 in $\mathbb{Z} + x\mathbb{Q}[x]$.

The constructions $A + xB[x]$ and $A + xI[x]$, specializations of the standard "$D + \mathfrak{M}$" construction, are gold mines for interesting (especially nonunique) factorizations [10] (note that it is assumed that $A \subseteq B$ are domains and in the second case that $I \subseteq A$ is an ideal). Indeed, there is no way that an $A+xB[x]$ construction can ever be a UFD unless $A = B$ and A is a UFD (in fact, unless $A = B$ and A is completely integrally closed, $A+xB[x]$ is not completely integrally closed). Also note that if I is a proper nonzero ideal of A, then x is almost integral over $A+xI[x]$ and hence $A + xI[x]$ is not a UFD either.

For the $A + xI[x]$ construction, assuming that A is an UFD is a very natural starting point. With respect to this condition, Gonzalez, Pellerin and Roberts proved the following nice theorem [30].

Theorem 7.1. *Let A be a UFD and $R := A + xI[x]$ with I a proper nonzero ideal of A. Then R is an HFD if and only if I is a prime ideal of A.*

Actually much more is shown in this paper. They show, in fact, that R is atomic and that $\rho(R)$ is finite if and only if $I = \mathfrak{P}_1 \cdots \mathfrak{P}_k$ with the $\mathfrak{P}_i$'s being noncomparable primes with at most one of them nonprincipal (and if this is the case then $\rho(R) = k$).

This result has also been extended to the case where A is a Dedekind domain. It has been shown that if $|\mathrm{Cl}(A)| = \infty$ then $\rho(R) = \infty$. In the case where $|\mathrm{Cl}(A)| < \infty$ and I is a product of k distinct principal primes, we have the following from [37].

Theorem 7.2. *Let A be a Dedekind domain with finite class group and let $I \subseteq A$ be the product of k distinct principal primes. Then $\rho(A + xI[x]) = k + \frac{D(Cl(A))}{2}$, where $D(Cl(A))$ is the Davenport constant of $Cl(A)$.*

Here we declare that the Davenport constant is 0 if the class group is trivial.

We now present some recent advances in the spirit of the work of Gonzalez, Pellerin, and Roberts. This next result can be found in [23] and is an extension of some of the work of Kim [33].

Theorem 7.3. *If K is a field and B is any domain containing K then $K + xB[x]$ is an HFD if and only if B is integrally closed.*

We now look at the $A + xI[x]$ construction.

Question 7.4. *If $K \subseteq B$ is an extension of domains with K a field and B a UFD, and $\mathfrak{P} \subseteq B$ is a prime ideal, then is $K + x\mathfrak{P}[x]$ an HFD?*

To help in this, we record the following result [23].

Theorem 7.5. *Let $A \subseteq B$ be an extension of domains with A a UFD and let $I \subseteq B$ be a proper ideal. If I contains an irreducible element $\pi \in B$ such that π^2 has no nontrivial factor in A, then $A + xI[x]$ is not an HFD.*

Proof.

$$(\pi x^2)(\pi^2 x) = (\pi x)(\pi x)(\pi x).$$

∎

Corollary 7.6. *Let $K \subseteq B$ be an extension of domains with K a field and let $I \subseteq B$ be a proper ideal. If I contains an irreducible element of B, then $K + xI[x]$ is not an HFD.*

This answers the question raised above. Once again, we see the usefulness of the ideal theoretic characterization of UFD. This time, the fact that the prime ideal $xI[x] \subseteq K + xI[x]$ contains no prime element helps us to do something "bad".

8 Integral closure

As UFDs are (completely) integrally closed, a very natural question is "are HFDs integrally closed?" Earlier examples quickly show that this is not the case; for example, we have seen the concrete example of the ring $\mathbb{Z}[\sqrt{-3}]$ which is a non-integrally closed HFD. Along this line of thought, a natural next question is, "is the integral closure of an HFD still an HFD?" This question was answered in the negative by the present author in [21]. We will outline this result (as the construction is somewhat intricate). But we will note that, in some sense, this counterexample is a bit unsatisfactory. More specifically, the reason why the integral closure of the HFD presented in [21] fails to have the half-factorial property is that it fails to be *atomic*. From the point of view of the current accepted definition of HFD, this example is a bona fide counterexample, but only because of the loss of atomicity. As one will see from the outline of the construction, it turns out that any two irreducible factorizations in the integral closure of this domain are of the same length. (So from the point of view of the original discarded definition of Zaks, the integral closure of this HFD is again an HFD).

We will outline this example in a couple of steps and give the briefest idea of how it works. This construction will deliver a specific example of an HFD whose integral closure does not have the half-factorial property, but it should be noted that this construction can be generalized (for example, the choice of the value group and residue fields are only for computational convenience).

1. We begin with the semigroup ring $\mathbb{F}_2[x; \mathbb{Q}^+]$, and we denote the maximal ideal generated by the "monomials" by $\mathfrak{M}$ (i.e., $\mathfrak{M}$ is the ideal generated by the set $\{x^\alpha\}_{\alpha \in \mathbb{Q}^+}$).

2. We now form the localization $R := \mathbb{F}_2[x; \mathbb{Q}^+]_{\mathfrak{M}}$. This localization is a $1-$dimensional nondiscrete valuation domain with value group $\mathbb{Q}$ and residue field $\mathbb{F}_2$.

3. Abusing the notation, we let $\mathfrak{M} \subseteq R$ be the maximal ideal and consider $R_1 := \mathbb{F}_2 + t\mathfrak{M}[t]$ where t is a polynomial indeterminate. It is worth noting at this step that the element $t \notin R_1$; additionally for all $\alpha \in \mathbb{Q}^+$, $x^\alpha \notin R_1$. It is also worth noting that R_1 is integrally closed. Indeed, its integral closure is certainly contained in $\mathbb{F}_2 + tR[t]$ since $R[t]$ is integrally closed and $\mathbb{F}_2 + tR[t]$ is integrally closed in $R[t]$ (the integral closure of $\mathbb{F}_2$ in R is $\mathbb{F}_2$). It is now easily seen that no element of $(\mathbb{F}_2 + tR[t]) \setminus (\mathbb{F}_2 + t\mathfrak{M}[t])$ is integral over $\mathbb{F}_2 + t\mathfrak{M}[t]$.

4. In this step, we simplify by localizing at the ideal $t\mathfrak{M}[t]$. The effect of this is to get rid of irreducible "polynomials" that may crop up in the ring R_1 and

clarifies factorization behavior. We let $R_2 := (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}$ and note that this is an HFD. Indeed, if $\pi \in R_2$ is irreducible, then $\pi \in t\mathfrak{M}[t]R_2$, and this observation allows us to conclude that every irreducible in R_2 can be written in the form

$$u(x^{\alpha_1}t + \epsilon_2 x^{\alpha_2}t^2 + \cdots + \epsilon_n x^{\alpha_n}t^n)$$

with each u a unit in R_2, $\epsilon_i \in U(R_2)\bigcup\{0\}$, and each $\alpha_i \in \mathbb{Q}^+$.

At this step it is fairly straightforward to verify that R_2 is an HFD. Indeed, a general nonzero nonunit in R_2 assumes the form (with notation as above)

$$u(x^{\alpha_0}t^m + \epsilon_1 x^{\alpha_1}t^{m+1} + \cdots + \epsilon_k x^{\alpha_k}t^{m+k}).$$

In this form it is easy to see via a "degree argument" that the above element has at most m irreducible factors (since no rational power of x is an element of R_2). The fact that if $m > 1$, the above element actually has a factorization with m elements follows from the fact that the original valuation domain from earlier in this construction was nondiscrete. Note that if $\alpha = \min(\alpha_0, \alpha_1, \cdots, \alpha_k)$, then we can factor the above element

$$\begin{aligned} &u(x^{\alpha_0}t^m + \epsilon_1 x^{\alpha_1}t^{m+1} + \cdots + \epsilon_k x^{\alpha_k}t^{m+k}) \\ &= u(x^{\frac{\alpha}{2(m-1)}}t)^{m-1}(x^{\alpha_0 - \frac{\alpha}{2}}t + \epsilon_1 x^{\alpha_1 - \frac{\alpha}{2}}t^2 + \cdots + \epsilon_k x^{\alpha_k - \frac{\alpha}{2}}t^{k+1}). \end{aligned}$$

Of course this element may have many factorizations, but the above shows that each has length m.

5. The problem that we encounter now is that the domain from the previous step, R_2, is integrally closed, as it is a localization of the integrally closed domain, R_1. In particular, the strategy is to make the elements $\{x\alpha\}_{\alpha\in\mathbb{Q}^+}$ integral and to ensure that the integral closure of our target domain is not atomic. To continue our construction we wish to adjoin elements to R_2 that make this happen (while not destroying the half-factorial property). With this in mind, we consider $R_3 := (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}[x+t]$, obtained by adjoining the element $x+t$ of the quotient field of R_2 to R_2.

   In the ring R_3 there are two prime ideals worth noting. The first is the prime ideal $\mathfrak{P} = (x+t)$ (it is fairly easy to show that $x+t$ is a prime element of R_3) and second is the prime ideal Γ which is the extension of the maximal ideal $t\mathfrak{M}[t]R_2$ to our new domain R_3.

6. We now wish to, once again, simplify for the purposes of factorization. We let $R_4 := (\mathbb{F}_2 + t\mathfrak{M}[t])_{t\mathfrak{M}[t]}[x+t]_{(\mathfrak{P}\bigcup\Gamma)^c}$. By construction, the only prime ideals of R_4 are (the extensions of) $\mathfrak{P}$ and Γ. A simple computation shows that the only nonunits of R_4 are of the form $(x+t)^n h$ where h can be thought of as an element of R_2 (up to units). If we consider any irreducible factorization of $(x+t)^n h$, we note first that since $x+t$ is a prime element of R_4, we can ignore the effect of the $(x+t)^n$ factors. An argument similar to the proof that any nonunit in R_2 has the half-factorial property (by counting the multiplicity of t) applies here and so R_4 is an HFD.

7. To see that the integral closure of R_4 is not an HFD (more specifically, not atomic) note that

$$x^{\frac{1}{n}} = \frac{tx^{1+\frac{1}{n}}}{tx}$$

and so $x^{\frac{1}{n}}$ is in the quotient field of R_4. Also note that

$$(x^{\frac{1}{n}})^{2n} + (x+t)(x^{\frac{1}{n}})^n + xt = 0$$

and so $x^{\frac{1}{n}}$ is a root of the monic polynomial $Y^{2n} + Y^n(x+t) + xt \in R_4[Y]$. It is easy to see that the existence of the factorizations

$$x = (x^{\frac{1}{n}})^n$$

for all n deny the atomicity of the integral closure of R_4.

Purists (those who would like to follow the original concept of HFDs from Zaks) will be understandably displeased with this example, for, in a certain sense, it is cheating. Along lines similar to the above, it can be shown that the integral closure of R_4 contains every (positive) rational power of x and every positive integral power of t and hence is a localization of a polynomial ring over a (nondiscrete) valuation domain. As a consequence, it can be shown that any two irreducible factorizations of the same element are of equal length.

The question still remains as to whether or not the integral closure of an HFD is again an HFD if the integral closure remains atomic.

9 Orders in rings of integers

Despite the title of this paper, we take a step back and look at the dual notion of underrings of half-factorial domains (here, by "underring" of a ring T, we mean a ring $R \subseteq T$ such that R and T share the same quotient field). Much of the study here is motivated by questions concerning orders in rings of algebraic integers. We briefly recall that an order in a ring of integers is a ring R such that the integral closure $\overline{R}$ is the full ring of algebraic integers.

The study of the factorization behavior is well-motivated by questions in algebraic number theory and is important in a number of applications including representation of integers by quadratic forms. For an interesting example of the ramifications of the half-factorial property with respect to the representation of integers by quadratic forms applied to finite geometry, see [24] for an example.

Much of the theory of factorization has its roots (and is still motivated by) questions from number theory. In particular, Carlitz' paper was motivated by a number-theoretic question of Narkiewicz. Carlitz' result gave an interpretation of the half-factorial property in terms of the class group, which is a subject that has been of interest in algebraic number theory for a very long time.

In the spirit of factorization and the class group, here is a very famous open conjecture due to Gauss that has resisted solution for quite some time.

Conjecture 9.1. *There is an infinite number of real quadratic UFDs.*

Despite the apparent difficulty of Gauss' conjecture, we offer another conjecture along these lines.

Conjecture 9.2. *There is an infinite number of real quadratic HFDs that are not UFDs.*

The curious reader may at this juncture wonder about the status of the questions for the imaginary case. In light of the fact that the Carlitz result allows us to completely determine the half-factorial property via the class number (with class number one being a UFD and class number 2 being a non-trivial HFD), we present a couple of results that answer this question. The following result is due to H. Stark [44]

Theorem 9.3. *Let R be the ring of integers of an imaginary quadratic field. Then $|Cl(R)| = 1$ if and only if the field discriminant is* $-3, -4, -7, -8, -11, -19, -43, -67$*, or* -163.

The proof of the fact that the rings of integers corresponding to the field discriminants listed above are all UFDs is fairly straightforward. The converse of the theorem was only completely solved in the late 1960's, when Stark showed that there was no tenth imaginary quadratic field.

A result from analytic number theory states that there are only finitely many totally complex abelian extensions of $\mathbb{Q}$ of given degree and class number ([35]). This makes the existence of only finitely many imaginary quadratic HFDs immediate. Due to another result of Stark [45] we can list all imaginary quadratic HFDs that are rings of integers.

Theorem 9.4. *Let R be the ring of integers of an imaginary quadratic field. Then $|Cl(R)| = 2$ if and only if the field discriminant is* $-15, -20, -24, -35, -40, -51, -52, -88, -91, -115, -123, -148, -187, -232, -235, -267, -403$*, or* -427.

It is interesting to note that before this list was complete, an initial upper bound was set for the size of the discriminant of imaginary quadratic fields K with the class number equal to 2. This bound was $|d(K)| < 10^{1030}$. Stark's paper [45] found all of the fields, and it is interesting to note that, in fact, $|d(K)| \leq 427$.

There is a striking difference to consider when looking for HFDs that occur as orders in rings of algebraic integers, for unlike UFDs, HFDs do not have to be integrally closed. UFDs that occur as orders must occur as the full ring of algebraic integers (and in some sense, this may make them easier to find, since the tools generally used in this vein become more unwieldy when one leaves the assumption "integrally closed" behind). The upshot of all this is that, in searching for HFDs in rings of integers, we do not have to restrict to the full ring of integers; we can consider proper orders.

When considering which orders may contain HFDs, the following theorem is useful. It demonstrates that one can restrict the search for HFDs to orders inside rings of integers which are HFDs. It is worth noting that the following theorem is a positive answer the question, "Is the integral closure of an HFD again an HFD?" for number theorists.

Theorem 9.5. *Let R be an order in a ring of algebraic integers with integral closure $\overline{R}$. If R is an HFD, then $\overline{R}$ is an HFD.*

If the search for HFDs in imaginary quadratic fields is augmented to include the non-integrally closed case, we are done quickly. A proof of the following theorem using norms can be found in [19].

Theorem 9.6. *The ring* $\mathbb{Z}[\sqrt{-3}]$ *is the unique non-integrally closed imaginary quadratic HFD.*

Since we know that any non-integrally closed HFD that is an order in an imaginary quadratic field must appear as a subring of one of the 27 imaginary HFDs (the 9 UFDs and 18 nontrivial HFDs listed above), we could prove this theorem by restricting to searching for subrings of these 27 rings of integers. It is of interest to note, however, that the above theorem was known before it was known that HFDs as orders must be subrings of integrally closed HFDs (see [19] for a direct result or [31] for at least an implicit one). The proof of this theorem in [19] utilizes the norm, and since in the case of imaginary quadratic fields, the norm is positive definite, the proof is rather quick and clean (it basically depends only upon representing certain integers via the norm form). But it should be noted that although perhaps elegant in this case, generalizing this technique to more general fields (even real quadratic fields) is fraught with difficulties because of the difficulty determining integer representations when the form may take on negative values (and it also should be pointed out that even for relatively small degree fields, the norm form becomes quite large and hard to handle very quickly).

Despite the computational difficulties inherent in actually applying the norm, the above discussion and results beg the question as to how much information can be obtained by looking at factorizations from the point of view of norms. We begin by recalling a concept known as elasticity.

Definition 9.7. *Let S be an atomic set (that is, every element of S can be factored into irreducible elements of S). We define the elasticity of S to be*

$$\rho(S) = sup\{\frac{n}{m} | \alpha_1\alpha_2\cdots\alpha_n = \beta_1\beta_2\cdots\beta_m\}$$

where α_i, β_j are irreducible elements of S for all i, j.

Basically, elasticity is a global measure of how long ratios of differing (yet equal) factorizations in S can be.

The definition that we give slightly generalizes the definition of elasticity of a domain given in [46] and studied in (among others) [19, 28, 29, 30]. Here is a theorem that has appeared in a paper by Valenza [46] and was improved by Narkiewicz [36].

Theorem 9.8. *Let R be a ring of algebraic integers. Then $\rho(R) = \frac{D}{2}$ where D is the Davenport constant of the class group of R if R is not a UFD (and if R is a UFD, then $\rho(R) = 1$).*

We remind the reader that the Davenport constant is the invariant of a finite abelian group G that is the smallest positive integer, n, such that any sequence of n (not necessarily distinct) elements of G has a subsequence that sums to 0. The Davenport constant has been studied rather extensively in the realm of pure group theory and has become a topic of much interest in commutative algebra because of its natural connection with factorization. There are many remaining open questions relating to the Davenport constant (e.g., "Is there a closed form representation of the Davenport constant?"). The answers to many of these questions would have far-reaching ramifications in factorization theory. For a more in-depth look at the theory of the Davenport constant and some of its applications, see [19, 13, 26]

We now produce a theorem that is useful in determining factorization behavior in rings of integers. This theorem is a hybrid of a couple of theorems that can be found in [16] and [20].

Theorem 9.9. *Let $F/\mathbb{Q}$ be a Galois extension, R the ring of algebraic integers of F, and S the multiplicative set of integral norms from R to $\mathbb{Z}$. Then R is an HFD if and only if $\rho(S) = 1$, and additionally, R is a UFD if and only if S has unique factorization into irreducibles.*

It should be noted that, in general, for Galois extensions we have that $\rho(R) \geq \rho(S)$. Perhaps this is not too surprising since one would expect that the set of norms would lose some of the factorization information that is contained in R. It is a subtle point that although $\rho(R) \geq \rho(S)$ in general, one cannot lose "too much" information in the sense that $\rho(R) = 1$ if and only if $\rho(S) = 1$.

We would also like to highlight that the "Galois" assumption is very necessary here. In the non-Galois case, an example of a non-Galois extension with normal closure having Galois group S_5 can be produced to show the failure of this theorem in general.

Mainly because of its more complicated unit structure, the real case cannot be disposed of as easily as the imaginary case. We supply the following conjecture.

Conjecture 9.10. *There are infinitely many real quadratic HFDs (even inside $\mathbb{Z}[\sqrt{2}]$).*

This result makes use of the "boundary map" which we define below.

Definition 9.11. *Let R be an HFD with quotient field K. We define the boundary map $\partial_R : K \setminus \{0\} \longrightarrow \mathbb{Z}$ via*

$$\partial_R(\alpha) = n - k$$

where $\alpha \in K \setminus \{0\}$ is of the form $\alpha = \frac{\pi_1\pi_2\cdots\pi_n}{\xi_1\xi_2\cdots\xi_k}$ with each $\pi_i, \xi_j \in Irr(R)$. In the case that $R = K$ we declare that $\partial_R \equiv 0$.

The boundary map is a homomorphism from the multiplicative group $K \setminus \{0\}$ to the additive group $\mathbb{Z}$. The definition of the boundary map assumes that the domain, R, is an HFD and this is a necessary (and sufficient) condition for the boundary map to be well-defined.

In Zaks' paper [48] a function called the *length function* was defined. A length function is a function from the nonzero nonunits of R to the natural numbers such that $f(st) = f(s) + f(t)$ for all s, t nonzero, nonunit elements of R. Of course, HFDs are precisely the domains which admit a well-defined length function such that $f(x) = 1$ for all irreducible elements $x \in R$. The boundary map may be thought of as a natural generalization of this length function to a homomorphism from the set of nonzero elements of the quotient field of R to the integers.

In [18], the boundary map proved to be a useful tool in showing that, in the case of orders in rings of algebraic integers, the integral closure of an HFD is again an HFD. Since the time of the paper [18], more work has been done on exploring properties of the boundary map and attempts have been made, with some success, to glean properties of overring behavior of HFDs. One recent avenue of interest is the so-called "boundary valuation domains" introduced by J. Maney.

Definition 9.12. *Let R be an HFD with quotient field K and boundary map ∂_R. R is called a boundary valuation domain if for all $\alpha \in K$ such that $\partial_R(\alpha) \neq 0$, either α or α^{-1} is an element of R.*

This definition was motivated by an attempt to understand the overring behavior of HFDs in the spirit of "ordinary" valuation domains. A good example of a boundary valuation domain is any Noetherian valuation domain, and a more general example is given by the construction $F + xK[[x]]$, where $F \subsetneq K$ are fields.

In [34], many interesting properties of the boundary valuation domains are explored, and insights into the overring behavior of HFDs are obtained. Additionally, boundary valuation domains are completely characterized using groups of divisibility. Recently, Maney has also shown that the class of boundary valuation domains coincides with the class of atomic pseudo-valuation domains.

We close this section by pointing to some interesting recent work along these lines. In a fairly recent sequence of papers [38, 39, 40, 41] M. Picavet-L'Hermitte has done some striking work (often utilizing t-closedness, seminormality, and the concept of weak factoriality) that has advanced the understanding of the factorization behavior in orders of rings of integers.

Finally, we leave the reader with a list of questions that the present author would like to see answered. This list of questions is by no means exhaustive (and is motivated by the present author's current tastes). There are myriad interesting open questions with respect to the structure and behavior of HFDs (and certainly many, many more in factorization theory in general). It seems clear that despite recent vigorous progress, factorization theory has very far to go.

Questions:

1. If $R[x]$ is an HFD, does it follow that $R[x, y]$ is an HFD? (Note: it is known that the answer to this is "yes" *if* if R is Noetherian.)
2. If $R[[x]]$ is an HFD, then is $R[[x, y]]$ an HFD? (Note: this one might be quite difficult; the analogous question for UFDs has been open for nearly 40 years [43].)
3. If the integral closure of an HFD is atomic, then is it an HFD?
4. Find a general ideal-theoretic characterization of HFDs (if it can be done)?
5. If R is an HFD, find general conditions for which $R[[x]]$ is an HFD.
6. Classify all (non-Noetherian) polynomial HFDs.
7. Under what conditions is an affine ring an HFD?
8. Is there an infinite number of (real) quadratic HFDs (inside $\mathbb{Z}[\sqrt{2}]$)?
9. Find a closed-form representation of the Davenport constant (if one exists).

10. Characterize the relationship between the elasticity of an order in a ring of integers and the elasticity of its set of norms. (Note: this is only partially known even in the case that the order is a full ring of integers and its quotient field is Galois over $\mathbb{Q}$.)

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