# PREP PROBLEMS

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## 1 Elementary divisibility problems

Let $R$ be an integral domain and $x, y, z \in R$.

1. (a) Show that $x|y \Leftrightarrow (y) \subseteq (x)$.

   (b) Show that $x|y$ and $y|x \Leftrightarrow (x) = (y) \Leftrightarrow y = ux$ for some $u \in U(R)$.

2. Show that $z = lcm(x, y) \Leftrightarrow (x) \cap (y) = (z)$.

3. Show that $(x, y) = (z) \Leftrightarrow z = gcd(x, y)$ and $z$ is a linear combination of $x$ and $y$.

4. Show that if $lcm(x, y)$ exists, then $gcd(x, y)$ exists, and moreover, $gcd(x, y)lcm(x, y) = xy$.

5. If $gcd(x, y)$ exists, does $lcm(x, y)$ necessarily exist?

6. Let $z \neq 0$. Show that if $gcd(xz, yz)$ exists, then $gcd(x, y)$ exists, and moreover, $gcd(xz, yz) = zgcd(x, y)$.

7. Show that if x is prime, then x is irreducible. Give examples to show that the converse is false.

8. Show that every two elements of $R$ have a gcd $\Leftrightarrow$ every two elements of $R$ have an lcm. Such an integral domain $R$ is called a *GCD-domain*.

9. Show that in a GCD-domain, every irreducible element is prime.

10. Show that $R$ is a UFD $\Leftrightarrow$ $R$ is atomic (i.e., every nonzero nonunit of $R$ is a product of irreducible elements) and a GCD-domain. Thus every irreducible element of a UFD is prime.

## 2 Prime and irreducible elements

1. Show that $\mathbf{Z} + X\mathbf{Q}[X]$ is not atomic.

2. Find the irreducible and prime elements of $\mathbf{Z} + X\mathbf{Q}[X]$.

3. Find an infinite family of prime elements $\{p_\alpha\}$ of $\mathbf{Z} + X\mathbf{Q}[X]$ such that $\cap(p_\alpha) \neq \{0\}$.

4. Let $K$ be proper subfield of a field $L$. Find the irreducible and prime elements of $K + XL[X]$.

5. Let K be a field. Find the irreducible and prime elements of $K[X^2, X^3]$.

6. Does $\mathbf{Q}[X; \mathbf{Q}_+]$ have any irreducible elements?

7. Does $\mathbf{R}[X; \mathbf{Q}_+]$ have any irreducible elements?

# 3 Monoid domain problems

Let $R$ be a commutative ring and $S$ an additive commutative monoid. Define the monoid ring $R[X; S] = \{\sum r_s X^s \mid r_s \in R, s \in S$, almost all $r_s = 0\}$ with $\sum a_s X^s + \sum b_s X^s = \sum (a_s + b_s) X^s$ and multiplication defined by $(a_s X^s)(b_t X^t) = a_s b_t X^{s+t}$ (and then extend using distributivity).

For example: $R[X; \mathbf{Z}_+] = R[X]$, $R[X; \mathbf{Z}] = R[X, X^{-1}]$, and $R[X; \{0, 2, 3, \ldots\}] = R[X^2, X^3]$.

1. Let $S$ be a submonoid of a torsionfree abelian group $G$. Show that $S$ is cancellative and torsionfree. (A monoid $S$ is *torsionfree* if $nx = ny$ for $x, y \in S$ and $n \geq 1 \Rightarrow x = y$.)

2. Let $S$ be a torsionfree commutative cancellative monoid. Show that $S$ is a submonoid of a torsionfree abelian group. (Thus $S$ is a submonoid of some $\oplus\mathbf{Q}$.)

3. Show that an abelian group can be totally ordered if and only it is torsionfree.

4. Let $G$ be an abelian group. Show that $R[X; G]$ is an integral domain $\Leftrightarrow$ $R$ is an integral domain and $G$ is torsionfree.

5. Show that $R[X; S]$ is an integral domain $\Leftrightarrow$ $R$ is an integral domain and $S$ is a torsionfree commutative cancellative monoid.

6. Let $R[X; S]$ be an integral domain. Show that $U(R[X; S]) = \{rX^s \mid r \in U(R),\ s \in U(S)\}$. (Here $U(S) = S \cap -S$ is the group of elements of $S$ with inverses.)

# 4 Localization problems

Let $R$ be an integral domain with quotient field $K$. Let $S \subseteq R \setminus \{0\}$ be a nonempty multiplicatively closed set. The *localization* of $R$ at $S$ is $R_S = \{r/s \in K \mid r \in R, s \in S\}$. Define $\bar{S} = \{x \in R \mid xy \in S$ for some $y \in R\}$. Clearly $S \subseteq \bar{S}$. We say that $S$ is *saturated* if $\bar{S} = S$.

1. Show that $R_S$ is a subring of $K$ containing $R$.
2. Show that $xy \in \bar{S} \Leftrightarrow x, y \in \bar{S}$. Thus $\bar{S}$ is multiplicatively closed and saturated.
3. Show that $R_{\bar{S}} = R_S$.
4. Let $P$ be a prime ideal of $R$ and let $S = R \setminus P$. Show that $\bar{S} = S$ and that $R_S$ is an integral domain with a unique maximal ideal $P_S = \{ p/s \mid p \in P,\ s \in R \setminus P \}$. This ring is denoted by $R_P$.
5. Let $\mathcal{P} = \{p_\alpha\}$ be a set of prime elements of $R$, and let $S = \{ up_1 \cdots p_n \mid u \in U(R),\ p_i \in \mathcal{P} \}$. We say that $S$ is *generated* by prime elements of $\mathcal{P}$.
   (a) Show that $S$ is a saturated multiplicatively closed set.
   (b) Suppose that $R$ is atomic. Show that $R_S$ is atomic and $R_S$ has no prime elements.
   (c) Show that $R_S$ satisfies ACCP (resp., is a BFD, an HFD, an FFD) if $R$ satisfies ACCP (resp., is a BFD, an HFD, an FFD).
6. Give an example of an integral domain $R$ which is not atomic and a multiplicatively closed subset $S$ of $R$ generated by prime elements of $R$ such that $R_S$ is a UFD (and hence is atomic, satisfies ACCP, is a BFD, an HFD, and an FFD).
7. Let $\mathcal{P} = \{p_\alpha\}$ be a set of prime elements of $R$ such that (i) $\cap(p_\alpha^n) = \{0\}$ for all $p_\alpha \in \mathcal{P}$ and (ii) $\cap(p_{\alpha_n}) = \{0\}$ for any (infinite) sequence of nonassociate primes $p_{\alpha_n}$ in $\mathcal{P}$, and let $S = \{ up_1 \cdots p_n \mid u \in U(R),\ p_i \in \mathcal{P} \}$. Show that $R$ is atomic (resp., satisfies ACCP, a BFD, an HFD, an FFD, a UFD) if $R_S$ is atomic (resp., satisfies ACCP, a BFD, an HFD, an FFD, a UFD).
8. Let $R = K[X; T]$, where $K$ is a field and $T = \{ q \in \mathbf{Q}_+ \mid q \geq 1 \} \cup \{0\}$ is an additive submonoid of $\mathbf{Q}_+$. Show that $R$ is a BFD, and hence $R$ satisfies ACCP and $R$ is atomic. Is $R$ an HFD or an FFD? Let $S = \{ X^q \mid 0 \neq q \in T \}$. Show that $R_S$ is not atomic, and hence $R$ does not satisfy ACCP and is not a BFD.

# 5 Group of divisibility problems

Let $R$ be an integral domain with quotient field $K$. Let $K^* = K \setminus \{0\}$, and let $U(R)$ be the group of units of $R$. Then $K^*$ is an abelian group under the usual multiplication, and $U(R)$ is a subgroup of $K^*$. The *group of divisibility of* $R$ is the multiplicative group $G(R) = K^*/U(R)$. For $x, y \in K^*$, we define $xU(R) \leq yU(R) \Leftrightarrow y/x \in R\, (\Leftrightarrow x|y \Leftrightarrow (y) \subseteq (x))$.

It is known ("The Krull-Kaplansky-Jaffard-Ohm Theorem") that if $G$ is a lattice ordered abelian group, then there is a Bézout domain $R$ with $G(R)$ order-isomorphic to $G$.

1. Show that $G(R)$ is a partially ordered abelian group under $\leq$. Note that $G(R)_+ = R^*/U(R)$.

2. Show that $G(R)$ is directed (i.e., for $x, y \in G(R)$, there is a $z \in G(R)$ with $x \leq z$ and $y \leq z$).

3. Show that $Prin(R) = \{ (x) \mid x \in K^* \}$ (partially ordered by reverse inclusion) and $G(R)$ are order-isomorphic (written $\cong_o$).

4. Show that $G(R)$ is totally ordered if and only if $R$ is a valuation domain. (Recall that $R$ is a *valuation domain* $\Leftrightarrow$ for all $x, y \in K$, either $x|y$ or $y|x$ ( $\Leftrightarrow$ the (principal)(fractional) ideals of $R$ are totally ordered under inclusion $\Leftrightarrow$ for all $x \in K^*$, either $x \in R$ or $x^{-1} \in R$).

5. Show that $G(R)$ is lattice ordered (i.e., any two elements of $G(R)$ have an inf (equivalently, a sup) in $G(R)$) $\Leftrightarrow$ $R$ is a GCD-domain.

6. Show that $G(R) \cong_o \oplus \mathbf{Z}$ (with the usual product order) $\Leftrightarrow$ $R$ is a UFD.

# 6 Length function problems

Let $R$ be an integral domain.

1. Show that $R$ is an BFD if and only if there is a function $f : R^* \longrightarrow \mathbf{Z}_+$ such that (i) $f(x) = 0 \Leftrightarrow x \in U(R)$, and (ii) $f(xy) \geq f(x) + f(y)$ for all $x, y \in R^*$.

2. Show that $R$ is an HFD $\Leftrightarrow$ there is a function $f : R^* \longrightarrow \mathbf{Z}_+$ such that (i) $f(x) = 0 \Leftrightarrow x \in U(R)$, (ii) $f(x) = 1 \Leftrightarrow x \in R$ is irreducible, and (iii) $f(xy) = f(x) + f(y)$ for all $x, y \in R^*$.

3. Show that $R$ is an HFD $\Leftrightarrow$ there is a function $f : R^* \longrightarrow \mathbf{Z}_+$ such that (i) $f(xy) = f(x) + f(y)$ for all $x, y \in R^*$, (ii) $f(x) = 1 \Leftrightarrow x \in R$ is irreducible, and (iii) $im f = \mathbf{Z}_+$.

4. Let $R \subseteq T$ be an extension of integral domains with $U(T) \cap R = U(R)$. Show that if $T$ is a BFD, then $R$ is a BFD. In particular, this holds when $R \subseteq T$ is an integral extension.

5. Show that $R[X]$ is a BFD $\Leftrightarrow$ $R$ is a BFD. This also holds for any family of indeterminates.

6. Show that $R[[X]]$ is a BFD $\Leftrightarrow$ $R$ is a BFD.

7. Show that a Krull domain is a BFD, and hence a Noetherian integral domain is a BFD. (Recall that the integral closure of a Noetherian integral domain is a Krull domain.)

## 7 Elasticty problems

1. Let $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$ be a sequence of subfields of a field $F = \cup F_n$, and let $R = \prod F_n X^n = \{ \sum a_n X^n \in F[[X]] \mid \text{each } a_n \in F_n \} \subseteq F[[X]]$.
   (a) Show that if $F_{N-1} \neq F_N = F$, then $\rho(R) = N$.
   (b) Show that if each $F_n$ is a proper subfield of $F$, then $\rho(R) = \infty$.
2. Let $R$ be a Dedekind domain.
   (a) Find the possible values for $\rho(R)$ when $Cl(R) = \mathbf{Z_4}$.
   (b) Find the possible values for $\rho(R)$ when $Cl(R) = \mathbf{Z_2} \oplus \mathbf{Z_2}$.
3. Show that there is a Dedekind domain $R$ with $Cl(R) = \mathbf{Z}$ and $1 < \rho(R) < \infty$.
4. Show that for every integer $n \geq 2$, there is a Dedekind HFD $R$ with $Cl(R) = \mathbf{Z_n}$.
5. Show that if $G$ is any finite abelian group, then there is a Dedekind HFD $R$ with $Cl(R) = G$.
6. Let $R = K[X;T]$, where $K$ is a field and $T = \{ q \in \mathbf{Q}_+ \mid q \geq 1 \} \cup \{0\}$. What can you say about $\rho(R)$?
7. Let $R$ be an atomic integral domain, and let $S$ be a multiplicatively closed subset of $R$ generated by prime elements of $R$. Show that $\rho(R_S) = \rho(R)$.

## 8 Pullback problems

Let $T$ be an integral domain with quotient field $L$, and let $M$ a nonzero maximal ideal of $T$. Let $K = T/M$ and $\pi : T \longrightarrow T/M = K$ be the natural surjection. Let $D$ be a subring of $K$ and $R = \pi^{-1}(D) \subseteq T$. Then $R$ is called a *pullback*.

A special case of the above is the *(generalized) "D + M" construction*. Let $T$ be an integral domain of the form $T = K + M$, where $K$ is a field and $M$ is a nonzero maximal ideal of $T$. Let $D$ be a subring of $K$. Then $R = D + M$ is a subring of $T = K + M$.

Some examples: (i) Let $K$ be a field, $T = K[X_1, \ldots, X_n]$ or $K[[X_1, \ldots, X_n]]$, $M = (X_1, \ldots, X_n)$, and $D$ a subring of $K$. Then $T = K + M$ and $R = D + M$. For $n = 1$, we have $R = D + XK[X]$ or $R = D + XK[[X]]$.

(ii) Let $T = \mathbf{R}[X]$ and $M = (X^2 + 1)$. Then $T/M = \mathbf{C}$, and let $D_1 = \mathbf{Z}$ and $D_2 = \mathbf{R}$. Find $\pi^{-1}(D_i)$ for $i = 1, 2$.

1. Show that $M$ is a prime ideal of $R$. Thus $R$ and $T$ have the same quotient field.
2. Show that $R = D + M$ is atomic (resp., satisfies ACCP, a BFD, an HFD, an FFD) $\Leftrightarrow$ $D$ is a field and $T = K + M$ is atomic (resp., satisfies ACCP, a BFD, an HFD, an FFD with $K^*/D^*$ finite).

3. Suppose that $T = K+M$ is atomic and $D$ is a field. Show that $\rho(D+M) = \rho(K+M)$.

4. Suppose that $T$ is quasilocal. Show that the pullback $R = \pi^{-1}(D)$ is atomic (resp., satisfies ACCP, a BFD, an HFD, an FFD) $\Leftrightarrow$ $D$ is a field and $T$ is atomic (resp., satisfies ACCP, a BFD, an HFD, an FFD with $U(T)/U(R)$ finite).

5. Suppose that $T$ is quasilocal and atomic and that $D$ is a field. Show that $\rho(R) = \rho(T)$, where $R = \pi^{-1}(D)$ is the pullback.